THE ANALOGUE OF HILBERT'S 1888 THEOREM FOR EVEN SYMMETRIC FORMS

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Abstract. Hilbert proved in 1888 that a positive semidefinite (psd) real form is a sum of squares (sos) of real forms if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (3, 4)$, where n is the number of variables and $2d$ the degree of the form. We study the analogue for even symmetric forms. We establish that an even symmetric n-ary $2d$-ic psd form is sos if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 4)_{n\geq 3}$ or $(n, 2d) = (3, 8)$.

1. Introduction

A real form (homogeneous polynomial) f is called *positive semidefinite* (psd) if it takes only non-negative values and it is called a *sum of squares* (sos) if there exist other forms h_j so that $f = h_1^2 + \cdots + h_k^2$. Let $\mathcal{P}_{n,2d}$ and $\Sigma_{n,2d}$ denote the cone of psd and sos n-ary $2d$-ic forms (i.e. forms of degree $2d$ in n variables) respectively.

In 1888, Hilbert [9] gave a celebrated theorem that characterizes the pairs $(n, 2d)$ for which every n-ary $2d$-ic psd form can be written as a sos of forms. It states that every n-ary $2d$-ic psd form is sos if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (3, 4)$. Hilbert demonstrated that $\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d}$ for $(n, 2d) = (4, 4), (3, 6)$, thus reducing the problem to these two basic cases.

Almost ninety years later, Choi and Lam [1] returned to this subject. In particular, they considered the question of when a symmetric psd form is sos. A form $f(x_1, \ldots, x_n)$ is called *symmetric* if $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$ for all $\sigma \in S_n$. As an analogue of Hilbert's approach, they reduced the problem to finding symmetric psd not sos n-ary $2d$-ics for the pairs $(n, 4)_{n\geq 4}$ and $(3, 6)$. They asserted the existence of psd not sos symmetric quartics in $n \geq 5$ variables; contingent on these examples, the answer is the same as that found by Hilbert. In [6], we constructed these quartic forms.

A form is *even symmetric* if it is symmetric and in each of its terms every variable has even degree. Let $S\mathcal{P}^e_{n,2d}$ and $S\Sigma^e_{n,2d}$ denote the set of even symmetric psd and even symmetric sos n-ary $2d$-ic forms respectively. Set $\Delta_{n,2d} := S\mathcal{P}^e_{n,2d} \backslash S\Sigma^e_{n,2d}$. In this paper, we investigate the following question:

$$Q(S^e) : \text{For what pairs } (n, 2d) \text{ is } \Delta_{n,2d} = \emptyset\ ?$$

The current answers to this question in the literature are $\Delta_{n,2d} = \emptyset$ if $n = 2, d = 1, (n, 2d) = (3, 4)$ by Hilbert's Theorem, $(n, 2d) = (3, 8)$ due to Harris [7], and $(n, 2d) = (n, 4)_{n\geq 4}$. The result $\Delta_{n,4} = \emptyset$ for $n \geq 4$ was attributed to Choi, Lam and Reznick in [7]; a proof can be found in [5, Proposition 4.1]. Further, $\Delta_{n,2d} \neq \emptyset$ for $(n, 2d) = (n, 6)_{n\geq 3}$ due to Choi, Lam and Reznick [3], for $(n, 2d) = (3, 10), (4, 8)$

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due to Harris [8] and for $(n, 2d) = (3, 6)$ due to Robinson [10]. Robinson's even symmetric psd not sos ternary sextic is the form

$$R(x, y, z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2.$$

Thus the answer to $Q(S^e)$ in the literature can be summarized by the following chart:

deg \ var	2	3	4	5	...
2	✓	✓	✓	✓	...
4	✓	✓	✓	✓	...
6	✓	×	×	×	...
8	✓	✓	×	o	o
10	✓	×	o	o	o
12	✓	o	o	o	o
14	✓	o	o	o	o
⋮	⋮	o	o	o	o

where, a tick (✓) denotes a positive answer to $Q(S^e)$, a cross (×) denotes a negative answer to $Q(S^e)$, and a circle (o) denotes "undetermined". Indeed to get a complete answer to $Q(S^e)$, we need to investigate the question in these remaining cases, namely $(n, 8)$ for $n \geq 5$, $(3, 2d)$ for $d \geq 6$ and $(n, 2d)$ for $n \geq 4, d \geq 5$.

Main Theorem. An even symmetric n-ary $2d$-ic psd form is sos if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 4)_{n\geq3}$ or $(n, 2d) = (3, 8)$.

In other words, every "o" in the chart can be replaced by "×".

The article is structured as follows. In Section 2, we develop the tools (Theorem 2.3 and Theorem 2.4) we need to prove our Main Theorem. These tools allow us to reduce to certain basic cases, in the same spirit as Hilbert and Choi-Lam. In Section 3 and Section 4 we resolve those basic cases by producing explicit examples for $(n, 2d); n \geq 4, d = 4, 5, 6$. We conclude Section 4 by interpreting even symmetric psd forms in terms of preorderings using our Main Theorem. Finally, for ease of reference we summarize our examples in Section 5.

2. Reduction to basic cases

The following Lemma will be used in Theorem 2.3.

Lemma 2.1. For $n \geq 3$, the even symmetric real forms

$$p_n(x_1, \ldots, x_n) = 4\sum_{j=1}^{n} x_j^4 - 17 \sum_{1\leq i<j\leq n} x_i^2x_j^2;$$

$$q_n(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j^6 + 3 \sum_{1\leq i\neq j\leq n} x_i^4x_j^2 - 100 \sum_{1\leq i<j<k\leq n} x_i^2x_j^2x_k^2$$

are irreducible over $\mathbb{R}$.

Proof. First observe that if a form g has a factorization

$$g(x_1, \ldots, x_n) = \prod_{r=1}^{u} f_r(x_1, \ldots, x_n),$$

then the same holds when $x_{k+1} = \cdots = x_n = 0$, hence it suffices to show that p_3 and q_3 are irreducible over $\mathbb{R}$. Second, observe that if (in addition) g is even and symmetric, then for all $\sigma \in S_n$ and choices of sign, $f_r(\pm_1 x_{\sigma_1}, \ldots, \pm_n x_{\sigma_n})$ is also a factor of g. We call distinct (non-proportional) forms of this kind *cousins* of f_r. If (in addition) f_r is irreducible, $\deg f_r = d$ and $\deg g = n$, then f_r can have at most n/d cousins.

If p_3 is reducible, then it has a factor of degree ≤ 2. Suppose that p_3 has a linear factor $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$. Upon setting $x_3 = 0$, we see that

$$\alpha_1 x_1 + \alpha_2 x_2 \mid 4x_1^4 - 17x_1^2x_2^2 + 4x_2^4 = (x_1 + 2x_2)(x_1 - 2x_2)(2x_1 + x_2)(2x_1 - x_2),$$

so $\alpha_2/\alpha_1 \in \{\pm 1/2, \pm 2\}$. Similarly, $\alpha_3/\alpha_2 \in \{\pm 1/2, \pm 2\}$, so $\alpha_3/\alpha_1 \in \{\pm 1/4, \pm 1, \pm 4\}$, which contradicts $\alpha_3/\alpha_1 \in \{\pm 1/2, \pm 2\}$. It follows that p_3 has no linear factors.

Suppose p_3 has a quadratic (irreducible) factor $f = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + \ldots$. If it is not true that $\alpha_1 = \alpha_2 = \alpha_3$, then by permuting variables, f has at least $3 > 4/2$ cousins. Thus $\alpha_1 = \alpha_2 = \alpha_3$, and by scaling we may assume the common value is 2. The binary quartic $4x_1^4 - 17x_1^2x_2^2 + 4x_2^4$ has six quadratic factors, found by taking pairs of linear factors as above. Of these, the ones in which $\alpha_1 = \alpha_2$ are $2x_1^2 \pm 5x_1x_2 + 2x_2^2$. It follows that, more generally, the coefficient of x_ix_j is ± 5 and that

$$f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2 + \pm_{12}(5x_1x_2) + \pm_{13}(5x_1x_3) + \pm_{23}(5x_2x_3).$$

Regardless of the initial choice of signs, making the single sign changes $x_i \mapsto -x_i$ for $i = 1, 2, 3$ shows that f has 4 cousins, which again is too many. Therefore, we may conclude that p_3 is irreducible.

We turn to q_3 and first observe that

$$q_3(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)^3 - 106x_1^2x_2^2x_3^2.$$

Suppose now that q_3 is reducible, so it has at least one factor of degree ≤ 3, and let f be such a factor of q_3. Once again, we set $x_3 = 0$ and observe that

$$f(x_1, x_2, 0) \mid q_3(x_1, x_2, 0) = (x_1^2 + x_2^2)^3.$$

Since $x_1^2 + x_2^2$ is irreducible over $\mathbb{R}$, we conclude that $\deg f = 2$ and $f(x_1, x_2) = \lambda(x_1^2 + x_2^2)$. Writing

$$f(x_1, x_2, x_3) = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + \sum_{1 \leq i < j \leq 3} \beta_{ij} x_i x_j,$$

we see from the foregoing that $\alpha_1 = \alpha_2$ and $\beta_{12} = 0$. By setting $x_2 = 0$ and $x_1 = 0$ in turn, we see that the α_i's are equal and $\beta_{ij} = 0$, so f is a multiple of $x_1^2 + x_2^2 + x_3^2$. But since $x_1^2x_2^2x_3^2$ is not a multiple of $x_1^2 + x_2^2 + x_3^2$, f cannot divide q_3, completing the proof. □

Lemma 2.2. Let f be a psd not sos n-ary $2d$-ic form and p an irreducible indefinite form of degree r in $\mathbb{R}[x_1, \ldots, x_n]$. Then the n-ary $(2d + 2r)$-ic form $p^2 f$ is also psd not sos.

Proof. See [6, Lemma 2.1]. □

Theorem 2.3. (**Degree Jumping Principle**) Suppose $f \in \Delta_{n,2d}$ for $n \geq 3$, then

1. for any integer $r \geq 2$, the form $f p_n^{2a} q_n^{2b} \in \Delta_{n,2d+4r}$, where $r = 2a + 3b$; $a, b \in \mathbb{Z}_+$, and p_n, q_n are as defined in Lemma 2.1;
2. $(x_1 \ldots x_n)^2 f \in \Delta_{n,2d+2n}$.

Proof. 1. For $r \in \mathbb{Z}_+, r \geq 2$, there exists non-negative $a, b \in \mathbb{Z}$ such that $r = 2a+3b$. Since $fp_n^{2a}q_n^{2b}$ is a product of even symmetric forms, it is even and symmetric; since it is a product of psd forms, it is psd. Thus we have $fp_n^{2a}q_n^{2b} \in S\mathcal{P}^e_{n,2d+4r}$. Since p_n and q_n are indefinite and irreducible forms by Lemma 2.1, we get $fp_n^2 \in \Delta_{n,2d+8}$ and $fq_n^2 \in \Delta_{n,2d+12}$ by Lemma 2.2. Finally, by repeating this argument we get $fp_n^{2a}q_n^{2b} \in \Delta_{n,2d+4r}$.

2. Taking $p = x_i$ in turn for each $1 \leq i \leq n$, the assertion follows by Lemma 2.2. □

Theorem 2.4. (Reduction to Basic Cases) If $\Delta_{n,2d} \neq \emptyset$ for $(n, 8)_{n\geq 4}, (n, 10)_{n\geq 3}$ and $(n, 12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \emptyset$ for $(n, 2d)_{n\geq 3, d\geq 7}$.

Proof. For $n = 3$, the basic examples are $R(x, y, z) \in \Delta_{3,6}$ (by Robinson [10]), several examples in $\Delta_{3,10}$ (by Harris [7]) and $p_3^2 R(x, y, z) \in \Delta_{3,14}$ (by Theorem 2.3 (1)). Every even integer ≥ 12 can be written as $6 + 6k, 10 + 6k$ or $14 + 6k$, $k \geq 0$, and so by Theorem 2.3 (2), $\Delta_{3,2d}$ is non-empty for $2d \geq 6$, $2d \neq 8$.

For $n \geq 4$, $\Delta_{n,6} \neq \emptyset$ (by Choi, Lam, Reznick [3]). We shall show in Sections 3 and 4 that $\Delta_{n,8}, \Delta_{n,10}, \Delta_{n,12}$ are non-empty. Every even integer ≥ 14 can be written as $6+8k, 8+8k, 10+8k$ or $12+8k$ and so, given our claimed examples, by Theorem 2.3, $\Delta_{n,2d}$ is non-empty for $n \geq 4$, $2d \geq 6$. □

In order to find psd not sos even symmetric n-ary octics, psd not sos even symmetric n-ary decics and psd not sos even symmetric n-ary dodecics for $n \geq 4$, we first recall the following theorems which will be particularly useful in proving the main results of Sections 3 and 4.

Theorem 2.5. Suppose $p = \sum_{i=1}^{r} h_i^2$ is an even sos form. Then we may write $p = \sum_{j=1}^{s} q_j^2$, where each form q_j^2 is even. In particular, $q_j(\underline{x}) = \sum c_j(\alpha)\underline{x}^\alpha$, where the sum is taken over α's in one congruence class mod 2 component-wise.

Proof. See [3, Theorem 4.1]. □

Theorem 2.6. A symmetric n-ary quartic f is psd if and only if $f(\underline{x}) \geq 0$ for every $\underline{x} \in \mathbb{R}^n$ with at most two distinct coordinates (if $n \geq 4$).

Proof. This was originally proved in [2]; see [5, Corollary 3.11], [7, Section 2]. □

Theorem 2.7. (i) For odd $2m + 1 \geq 5$, the symmetric $2m + 1$-ary quartic

$$L_{2m+1}(\underline{x}) := m(m + 1) \sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2$$

is psd not sos.

(ii) For $2m \geq 4$, the symmetric $2m$-ary quartic

$$C_{2m}(x_1, \ldots, x_{2m}) := L_{2m+1}(x_1, \ldots, x_{2m}, 0)$$

is psd not sos.

Proof. See [6, Theorems 2.8, 2.9]. □

Theorem 2.8. For an integer $r \geq 1$, let $M_r = M_r(x_1, \ldots, x_n) := x_1^r + \cdots + x_n^r$. For reals a, b, c, the sextic $p = aM_2^3 + bM_2M_4 + cM_6$ is psd if and only if $at^2 + bt + c \geq 0$ for $t \in \{1, 2, \ldots n\}$ and sos if and only if $at^2 + bt + c \geq 0$ for $t \in \{1\} \cup [2, n]$.

Proof. See [3, Theorems 3.7, 4.25]. □

Observation 2.9. Let v_t denote any n-tuple with t components equal to 1 and $n-t$ components equal to zero. Then $M_r(v_t) = t$, so $p(v_t) = t(at^2 + bt + c)$. It will be useful in the proofs of Theorems 3.1, 4.1 and 4.4 to let $v_t(a_1, \dots, a_t)$ denote the particular v_t with 1's in positions $a_1, \dots, a_t$.

3. Psd not sos even symmetric n-ary octics for $n \geq 4$

It follows from Theorem 2.7 that for $m \geq 2$,

$$G_{2m+1}(x_1, \dots, x_{2m+1}) := L_{2m+1}(x_1^2, \dots, x_{2m+1}^2) \in S\mathcal{P}^e_{2m+1,8};$$
$$D_{2m}(x_1, \dots, x_{2m}) := G_{2m+1}(x_1, \dots, x_{2m}, 0) \in S\mathcal{P}^e_{2m,8}.$$

We showed in [6] that $G_{2m+1}(\underline{x}) = 0$ for those $\underline{x} \in \mathbb{R}^{2m+1}$ which are a permutation of $m+1$ r's and m s's for $(r, s) \in \mathbb{R}^2$, so that $D_{2m}(\underline{x}) = 0$, projectively, at any v_m or v_{m+1}.

Theorem 3.1. For $m \geq 2$, $D_{2m} \in \Delta_{2m,8}$ and $G_{2m+1} \in \Delta_{2m+1,8}$.

Proof. We observe that $D_{2m}(v_1) > 0$; in fact, it is equal to $m(m+1)(2m) - (2m)^2 = 2m^2(m-1)$. Thus, the coefficient of x_i^8 in D_{2m} is positive. Suppose $D_{2m} = \sum h_t^2$. Then x_i^4 must appear with non-zero coefficient in at least one h_t. Since we may assume that h_t^2 is even (using Theorem 2.5), we must have

$$h_t = \sum_{i=1}^{n} a_i x_i^4 + \sum_{1 \leq i < j \leq n} b_{i,j} x_i^2 x_j^2.$$

Since $D_{2m}(v_m) = D_{2m}(v_{m+1}) = 0$, it follows that $h_t(v_m) = h_t(v_{m+1}) = 0$, and this holds for all permutations of v_m and v_{m+1}. Our goal is to show that these equations imply that $h_t = 0$, which will contradict the assumption that D_{2m} is sos. By symmetry, it suffices to prove that $a_i = 0$ for one choice of i.

To this end, let $y^{(1)} = v_m(1, \dots, m-1, 2m-1), y^{(2)} = v_m(1, \dots, m-1, 2m)$ and $y^{(3)} = v_{m+1}(1, \dots, m-1, 2m-1, 2m)$. Then

$$0 = h_t(y^{(1)}) = \sum_{i=1}^{m-1} a_i + a_{2m-1} + \sum_{1 \leq i < j \leq m-1} b_{i,j} + \sum_{i=1}^{m-1} b_{i,2m-1};$$
$$0 = h_t(y^{(2)}) = \sum_{i=1}^{m-1} a_i + a_{2m} + \sum_{1 \leq i < j \leq m-1} b_{i,j} + \sum_{i=1}^{m-1} b_{i,2m};$$
$$0 = h_t(y^{(3)}) = \sum_{i=1}^{m-1} a_i + a_{2m-1} + a_{2m} + \sum_{1 \leq i < j \leq m-1} b_{i,j} + \sum_{i=1}^{m-1} b_{i,2m-1} + \sum_{i=1}^{m-1} b_{i,2m} + b_{2m-1,2m}.$$

Taking the first equation plus the second minus the third yields

$$\sum_{i=1}^{m-1} a_i + \sum_{1 \leq i < j \leq m-1} b_{i,j} = b_{2m-1,2m}.$$

Since $m \geq 2$, $m-1 < 2m-2$; thus, the same argument implies that

$$\sum_{i=1}^{m-1} a_i + \sum_{1 \leq i < j \leq m-1} b_{i,j} = b_{2m-2,2m}.$$

That is, the coefficient of $x_{2m-1}^2 x_{2m}^2$ in h_t equals the coefficient of $x_{2m-2}^2 x_{2m}^2$, and so by symmetry, for all distinct i, j, k, ℓ, the coefficient of $x_i^2 x_j^2$ equals the coefficient of $x_i^2 x_k^2$, which equals the coefficient of $x_k^2 x_\ell^2$. Thus, for all $i \neq j$, $b_{i,j} = u$ for some u.

Subtracting the first from the second equation gives now $a_{2m-1} = a_{2m}$, and so for all i, $a_i = v$ for some v. Finally, our previous equations imply that

$$0 = mv + \binom{m}{2}u = (m+1)v + \binom{m+1}{2}u = 0$$

$$\implies -v = \frac{m-1}{2} \cdot u = \frac{m}{2} \cdot u \implies u = 0 \implies v = 0.$$

In other words, $h_t = 0$, establishing the contradiction.

Suppose now that G_{2m+1} were sos. Then

$$G_{2m+1} = \sum_{t=1}^{r} h_t^2 \implies D_{2m} = \sum_{t=1}^{r} h_t^2(x_1, \ldots, x_{2m}, 0),$$

a contradiction. Thus G_{2m+1} is not sos. □

Remark 3.2. It was asserted in [3] that the psd even symmetric n-ary octic

$$M_2\Big(M_2^3 - (2k+1)M_2M_4 + k(k+1)M_6\Big)$$

is not sos, provided $2 \leq k \leq n-2$. We prove this below for $k = 2$ and $n \geq 4$.

Theorem 3.3. For $n \geq 4$,

$$T_n(x_1, \ldots, x_n) = M_2\Big(M_2^3 - 5M_2M_4 + 6M_6\Big) \in \Delta_{n,8}.$$

Proof. Note that T_n is psd by Theorem 2.8. Suppose

$$T_n(x_1, \ldots, x_n) = \sum_{r=1}^{m} h_r^2(x_1, \ldots, x_n).$$

Then, $T_n(v_2) = T_n(v_3) = 0$ but $T_n(v_1) > 0$. In particular, the terms x_j^4 must appear on the right hand side. As in the proof of Theorem 3.1, these terms must appear in

$$\sum_{k=1}^{n} a_k x_k^4 + \sum_{1 \leq j < k \leq n} b_{jk} x_j^2 x_k^2,$$

which must vanish at every v_2 and every v_3. In particular, for $i < j < k$, we have

$$a_i + a_j + b_{ij} = 0,$$
$$a_i + a_k + b_{ik} = 0,$$
$$a_j + a_k + b_{jk} = 0,$$
$$a_i + a_j + a_k + b_{ij} + b_{ik} + b_{jk} = 0.$$

It easily follows that $a_i + a_j + a_k = 0$. Now assume i, j, k are distinct, but not necessarily increasing. Since $n \geq 4$, there is an unused index ℓ and we may conclude that $a_i + a_j + a_\ell = 0$. Hence $a_k = a_\ell$. Since these are arbitrary, we conclude that a_m is independent of m, and since $a_i + a_j + a_k = 0$, it follows that each $a_m = 0$, a contradiction. □

Remark 3.4. For $n = 3$, $M_2(M_2^3 - 5M_2M_4 + 6M_6) = 2M_2R$ is sos, see [10], or equation (7.4) in [3].

4. Psd not sos even symmetric n-ary decics and dodecics for $n \geq 4$

Theorem 4.1. For $n \geq 4$,

$$P_n(x_1, \dots, x_n) = (nM_4 - M_2^2)(M_2^3 - 5M_2M_4 + 6M_6) \in \Delta_{n,10}.$$

Proof. First recall that

$$nM_4 - M_2^2 = n\sum_{k=1}^{n} x_k^4 - \left(\sum_{k=1}^{n} x_k^2\right)^2 = \sum_{i<j}(x_i^2 - x_j^2)^2$$

is psd by Cauchy-Schwarz. The zero set is $(\pm1, \dots, \pm1)$.

Second, recall from Theorem 2.8 that the quadratic $t(at^2 + bt + c)$ gives the value of the sextic $aM_2^3 + bM_2M_4 + cM_6$ at an n-tuple v_t with t 1's and $n - t$ 0's. Since $t(t-2)(t-3) \geq 0$, this criterion is satisfied, and the second factor is also psd with zeros at v_2 and v_3.

It follows that P_n is psd and its coefficient of x_1^{10} is $(n-1)(1-5+6) > 0$. We show that P_n is not sos by showing that in any sos expression, x_1^{10} cannot occur.

Using Theorem 2.5, we see that if $P_n = \sum h_r^2$ and x_1^5 occurs in h_r, then

$$h_r = ax_1^5 + x_1^3\left(\sum_{k=2}^{n} b_k x_k^2\right) + x_1\left(\sum_{k=2}^{n} c_k x_k^4\right) + x_1\left(\sum_{2\leq j<k<n} d_{jk}x_j^2x_k^2\right).$$

Since $P_n(v_2(1, j)) = P_n(v_3(1, j, k)) = 0$ for all j, k, $2 \leq j < k \leq n$, it follows that $0 = h_r(v_2(1, j)) = h_r(v_3(1, j, k)) = 0$, and we have the equations

$$0 = a + b_j + c_j,$$
$$0 = a + b_j + b_k + c_j + c_k + d_{jk} = (a + b_j + c_j) + (a + b_k + c_k) + d_{jk} - a.$$

From these equations, we may conclude that for all $2 \leq j < k \leq n$,

$$b_k + c_k = -a, \qquad d_{jk} = a.$$

Finally, $P_n(v_n) = 0$, so $h_r(v_n) = 0$; that is,

$$0 = a + \sum_{k=2}^{n}(b_k + c_k) + \sum_{2\leq j<k<n} d_{jk} = a\left(1 - (n-1) + \binom{n-1}{2}\right) = a \cdot \frac{(n-2)(n-3)}{2}.$$

Thus, $a = 0$ and x_1^5 occurs in no h_r. This gives the contradiction. □

Remark 4.2. When $n = 3$, P_n is sos:
$$\begin{aligned} P_3 &= (3M_4 - M_2^2)(M_2^3 - 5M_2M_4 + 6M_6) \\ &= 4(x^4 + y^4 + z^4 - x^2y^2 - x^2z^2 - y^2z^2)R(x, y, z) \\ &= 4(x^2(x^2 - y^2)^2(x^2 - z^2)^2 + y^2(y^2 - x^2)^2(y^2 - z^2)^2 + z^2(z^2 - x^2)^2(z^2 - y^2)^2). \end{aligned}$$

Remark 4.3. We have also shown that for $m \geq 2$, $M_2G_{2m+1} \in \Delta_{2m+1,10}$. We shall discuss M_2G_{2m+1} and M_2D_{2m} in a future publication.

Theorem 4.4. For $n \geq 5$,

$$Q_n(x_1, \dots, x_n) = (M_2^3 - 5M_2M_4 + 6M_6)(M_2^3 - 7M_2M_4 + 12M_6) \in \Delta_{n,12}.$$

Proof. Since $(t-2)(t-3) \geq 0$ and $(t-3)(t-4) \geq 0$, both factors in Q_n are psd by Theorem 2.8. The first has zeros at every v_2 and v_3 and the second has zeros at every v_3 and v_4. But note that neither has a zero at v_1. In fact, the coefficient of x_1^6 in Q_n is $(1-5+6)(1-7+12) > 0$.

Suppose Q_n is sos and $Q_n = \sum f_k^2$. As before, assume the f_k^2's are even (using Theorem 2.5). Then $f_k(v_t) = 0$ for every v_t with $t = 2, 3, 4$. Since $Q_n(v_1) > 0$, there must be an f_k containing x_i^6, which will be itself even. To this end, suppose

$$f_k = \sum_{i=1}^{n} \alpha_i x_i^6 + \sum_{1 \le i \ne j \le n} \beta_{ij} x_i^4 x_j^2 + \sum_{1 \le i < j < k \le n} \gamma_{ijk} x_i^2 x_j^2 x_k^2.$$

For $i < j$, let $\mu_{ij} = \beta_{ij} + \beta_{ji}$. By evaluating at $v_2(i, j)$, we see that

$$0 = \alpha_i + \alpha_j + \beta_{ij} + \beta_{ji} = \alpha_i + \alpha_j + \mu_{ij} \implies \mu_{ij} = -\alpha_i - \alpha_j.$$

By evaluating at $v_3(i, j, k)$, we have

$$\begin{gathered} 0 = \alpha_i + \alpha_j + \alpha_k + \mu_{ij} + \mu_{ik} + \mu_{jk} + \gamma_{ijk} = \\ (\alpha_i + \alpha_j + \alpha_k) - 2(\alpha_i + \alpha_j + \alpha_k) + \gamma_{ijk} \implies \gamma_{ijk} = (\alpha_i + \alpha_j + \alpha_k). \end{gathered}$$

Finally, by evaluating at $v_4(i, j, k, \ell)$, we have

$$\begin{aligned} 0 &= \alpha_i + \alpha_j + \alpha_k + \alpha_\ell + \mu_{ij} + \mu_{ik} + \mu_{jk} + \mu_{i\ell} + \mu_{j\ell} + \mu_{k\ell} + \gamma_{ijk} + \gamma_{ij\ell} + \gamma_{ik\ell} + \gamma_{jk\ell} \\ &= (\alpha_i + \alpha_j + \alpha_k + \alpha_\ell)(1 - 3 + 3) \implies \alpha_i + \alpha_j + \alpha_k + \alpha_\ell = 0. \end{aligned}$$

In other words, the sum of any four distinct α_r's is 0. Since $n \ge 5$, there exists $m \in \{1, \dots, n\}$ different from i, j, k, ℓ and we have $\alpha_i + \alpha_j + \alpha_k + \alpha_m = 0$. Thus $\alpha_\ell = \alpha_m$, and since the choice of ℓ and m was arbitrary, we conclude that $\alpha_1 = \cdots = \alpha_n = \alpha$, so that $4\alpha = 0$ and thus the coefficient of x_i^6 in f_k must be zero, a contradiction. □

Remark 4.5. We have been unable to determine whether Q_3 and Q_4 are sos.

Theorem 4.6. For $n \ge 3$,

$$\begin{aligned} R_n(x_1, \dots, x_n) &= \frac{1}{12} \cdot (M_2^3 - 3M_2M_4 + 2M_6)(M_2^3 - 5M_2M_4 + 6M_6) \\ &= \left(\sum_{1 \le i < j < k \le n} x_i^2 x_j^2 x_k^2 \right) \left(\sum_{i=1}^{n} x_i^6 - \sum_{1 \le i \ne j \le n} x_i^4 x_j^2 + 3 \sum_{1 \le i < j < k \le n} x_i^2 x_j^2 x_k^2 \right) \in \Delta_{n,12}. \end{aligned}$$

Proof. Since $(t-1)(t-2) \ge 0$ and $(t-2)(t-3) \ge 0$, both factors in R_n are psd by Theorem 2.8. Moreover, the first factor implies that

$$R_n(t, u, 0, \dots, 0) = 0 \tag{4.1}$$

for all real t, u, and at all n-tuples which are permutations of $(t, u, 0, \dots, 0)$. We also have, for all t,

$$R_n(t, 1, 1, 0, \dots, 0) = t^2 \cdot \tfrac{1}{2} \cdot ((2+t^2)^3 - 5(2+t^2)(2+t^4) + 6(2+t^6)) = t^4(t^2-1)^2; \tag{4.2}$$

this also holds by symmetry at any permutation of $n-3$ 0's, two 1's and one t.

We first remark that if $n = 3$, $R_n(x, y, z) = x^2y^2z^2R(x, y, z)$; since R is not sos, the same holds for R multiplied by a product of squared linear factors. For $n \ge 4$, more work is necessary.

Suppose $R_n = \sum_r h_r^2$, so that $\deg h_r = 6$. Suppose as usual that each h_r^2 is even (using Theorem 2.5). It follows from equation (4.1) that for any such h_r, $h_r(t, u, 0, \dots, 0) = 0$, for all (t, u). If, say, the terms in h_r involving only $x_1^{6-k}x_2^k$ are $\sum_{k=0}^{6} a_k x_1^{6-k} x_2^k$, then $\sum a_k t^{6-k} u^k = 0$ for all t, u, which implies that $a_k = 0$. Proceeding similarly for all pairs of variables, we see that no monomial involving one or two variables can appear in any h_r.

For equation (4.2), let $\phi_r(t) = h_r(t, 1, 1, 0, \ldots, 0)$. We have

$$\sum_{r=1}^{w} \phi_r(t)^2 = t^4(t^2 - 1)^2.$$

Evaluation at $t = 0, 1, -1$ shows that $\phi_r(t) = t(t^2 - 1)\psi_r(t)$, so that

$$\sum_{r=1}^{w} \psi_r(t)^2 = t^2,$$

which in turn implies that $\psi_r(t) = \lambda_r t$ for some real λ_r. To recapitulate, we have

$$h_r(t, 1, 1, 0, \ldots, 0) = \lambda_r t^2(t^2 - 1), \tag{4.3}$$

and similar equations hold for all permutations of the variables.

Since $x_1^2 x_2^2 x_3^8$ appears in r_n with coefficient 1, it also appears in $\sum h_r^2$ with coefficient 1. Since no monomial occurs in any h_r with only two variables, it follows that $x_1 x_2 x_3^4$ must appear in at least one h_r, and since h_r^2 is even, all terms in h_r must be $x_1 x_2$ times an even quartic monomial. Further, we already know that $x_1^5 x_2, x_1^3 x_2^3, x_1 x_2^5$ do not occur. Thus

$$h_r(x_1, \ldots, x_n) = x_1 x_2 \left(\sum_{j=3}^{n} (a_j x_1^2 x_j^2 + b_j x_2^2 x_j^2 + c_j x_j^4) + \sum_{3 \le j < k \le n} d_{jk} x_j^2 x_k^2 \right).$$

Thus,

$$h_r(t, 1, 1, 0, \ldots, 0) = t(a_3 t^2 + b_3 + c_3),$$
$$h_r(1, t, 1, 0, \ldots, 0) = t(a_3 + b_3 t^2 + c_3).$$

In view of equation (4.3), both of these cubics must be identically zero, hence $a_3 = b_3 + c_3 = b_3 = a_3 + c_3 = 0$, and so, in particular, $c_3 = 0$. This means that $x_1 x_2 x_3^4$ does *not* appear in any h_r, establishing the contradiction. □

Proof of the Main Theorem.
Combine Theorems 2.4, 3.1, 4.1, 4.4 and 4.6. □

We now present an application of the Main Theorem to the interpretation of even symmetric psd forms in terms of preorderings. We briefly recall the necessary background. Let $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{R}[\underline{x}]$, and let

$$T_S := \left\{ \sum_{e_1, \ldots, e_s \in \{0,1\}} \sigma_{\underline{e}}\, g_1^{e_1} \ldots g_s^{e_s} \mid \sigma_{\underline{e}} \in \Sigma \mathbb{R}[\underline{x}]^2, \underline{e} = (e_1, \ldots, e_s) \right\}$$

be the associated finitely generated quadratic preordering, and

$$K_S := \{\underline{x} \in \mathbb{R}[\underline{x}] \mid g_1(\underline{x}) \ge 0, \ldots, g_s(\underline{x}) \ge 0\}$$

be the associated basic closed semi-algebraic set.

We recall the following result which follows from [11, Proposition 6.1]:

Proposition 4.7. Let S be a finite subset of $\mathbb{R}[\underline{x}]$, such that $\dim(K_S) \ge 3$. Then there exists a $g \in \mathbb{R}[\underline{x}]$ s.t. $g \ge 0$ on K_S but $g \notin T_S$.

In the concluding Remark 4.9, we investigate when can the form g of Proposition 4.7 be chosen to be symmetric. We set $S' := \{x_1, \ldots, x_n\}$ and $K_{S'} = \mathbb{R}_+^n$ (the positive orthant). We need the following relation between the preordering $T_{S'}$ and even sos forms, as verified in [4, Lemma 1]:

Lemma 4.8. Let $g \in \mathbb{R}[\underline{x}]$. Then $g(x_1^2, \dots, x_n^2)$ is sos if and only if $g \in T_{S'}$.

Remark 4.9. Let f be an even symmetric n-ary form of degree $2d$, and g be the n-ary symmetric form of degree d such that $g(x_1^2, \dots, x_n^2) = f(x_1, \dots, x_n)$. Clearly, f is psd if and only if g is nonnegative on $\mathbb{R}^n_+$. Moreover, by Lemma 4.8, $f \in \Sigma\mathbb{R}[\underline{x}]^2$ if and only if $g \in T_{S'}$. Applying our Main Theorem, we get that for $n \geq 3$, $d \geq 3$ and $(n, 2d) \neq (3, 8)$, there exists a *symmetric* n-ary d-ic form g such that $g \geq 0$ on $\mathbb{R}^n_+$ but $g \notin T_{S'}$.

5. A glossary of forms

For easy reference, we list the examples discussed in this paper.

$$L_{2m+1}(x_1, \dots, x_{2m+1}) := m(m+1) \sum_{i<j} (x_i - x_j)^4 - \Big(\sum_{i<j} (x_i - x_j)^2 \Big)^2, \quad \text{(Theorem 2.7)};$$

$$C_{2m}(x_1, \dots, x_{2m}) = L_{2m+1}(x_1, \dots, x_{2m}, 0), \quad \text{(Theorem 2.7)};$$

$$M_r(x_1, \dots, x_n) = x_1^r + \cdots + x_n^r, \quad \text{(Theorem 2.8)};$$

$$G_{2m+1}(x_1, \dots, x_{2m+1}) = L_{2m+1}(x_1^2, \dots, x_{2m+1}^2), \quad \text{(Section 3)};$$

$$D_{2m}(x_1, \dots, x_{2m}) = G_{2m+1}(x_1, \dots, x_{2m}, 0), \quad \text{(Section 3)};$$

$$T_n(x_1, \dots, x_n) = M_2\Big(M_2^3 - 5M_2M_4 + 6M_6\Big), \quad \text{(Theorem 3.3)};$$

$$P_n(x_1, \dots, x_n) = (nM_4 - M_2^2)(M_2^3 - 5M_2M_4 + 6M_6), \quad \text{(Theorem 4.1)};$$

$$Q_n(x_1, \dots, x_n) = (M_2^3 - 5M_2M_4 + 6M_6)(M_2^3 - 7M_2M_4 + 12M_6), \quad \text{(Theorem 4.4)};$$

$$R_n(x_1, \dots, x_n) = \frac{1}{12} \cdot (M_2^3 - 3M_2M_4 + 2M_6)(M_2^3 - 5M_2M_4 + 6M_6), \quad \text{(Theorem 4.6)}.$$

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