Notes on the Fibonacci numbers, Math 296, 1/29/01

Continuing the last notes, we had defined $\mathcal{F}$ to be the set of all sequences

$$x = (x_0, x_1, x_2, \dots), \quad x_n \in \mathbf{C}.$$

with the property that $x_n = x_{n-1} + x_{n-2}$ for $n \geq 2$. We had seen that $\mathcal{F}$ is a two-dimensional vector space and that the Fibonacci sequence F and its shift F' —

$$F = (0, 1, 1, 2, 3, 5, \dots) \qquad F' = (1, 1, 2, 3, 5, 8, \dots)$$

form a basis. In fact, for any sequence $x \in \mathcal{F}$, we have

$$x_n = x_0(F_{n+1} - F_n) + x_1 F_n = (-x_0 + x_1)F_n + x_0 F_{n+1}.$$

We concluded from this two addition formulas

(i) $$F_{n+k} = F_k F_{n+1} + F_{k+1} F_n - F_k F_n;$$

(ii) $$F_{n+k} = F_k F_{n-1} + F_{k+1} F_n.$$

The understanding here is that $n \geq 1$ so that we don't have a negative index.

If we set $k = n$ and $k = n-1$ in (ii) (a general technique one might call "diagonalizing the parameters"), we get formulas of independent interest:

(iii) $$F_{2n} = F_n(F_{n+1} + F_{n-1}) = 2F_n F_{n+1} - F_n^2;$$

(iv) $$F_{2n-1} = F_{n-1}^2 + F_n^2.$$

It follows from (iii) that $F_n \mid F_{2n}$ for all n, and the quotient is also in $\mathcal{F}$. If we now set $k = 2n$ in (ii), we get

$$F_{3n} = F_{2n} F_{n-1} + F_{2n+1} F_n,$$

so that $F_n \mid F_{3n}$ for all n. You may now be able to see an inductive procedure to prove that $F_n \mid F_{kn}$ for all $k \geq 1$, and we will soon explore the behavior of F_{3n}/F_n, etc.

Another possible direction is this: suppose we put $(k, n) \mapsto (n, n-1)$ in (ii) and compare it with (iii). We have

$$F_{2n} = F_{n+1} F_n + F_n F_{n-1}, \qquad F_{2n-1} = F_{n+1} F_{n-1} + F_n F_{n-2}.$$

This suggests that sequences like $F_{n+1} F_{n-k} + F_n F_{n-(k+1)}$ might have some interesting properties. You also might want to explore such expressions as $F_{n+1}^2 - F_n^2$, $F_{n+2} \pm F_n^2$, etc.

A third direction might be this: let $G_n = F_{2n}$. Is there a recurrence satisfied by G_n? (The first few are 0,1,3,8,21,55...) We'll see in the next set of notes that there's an easy and almost no-work way to show that $G_n = 3G_{n-1} - G_{n-2}$.

Back to the main train of thought. Suppose $\lambda \in \mathbf{C}$ (we'll see why we need to consider complex numbers in the next notes), and let

$$p_\lambda = (1, \lambda, \lambda^2, \dots).$$

When is $p_\lambda \in \mathcal{F}$? This will occur when the recurrence is satisfied; that is, when for all $n \geq 2$,

$$\lambda^n = \lambda^{n-1} + \lambda^{n-2} \implies \lambda^{n-2}(\lambda^2 - \lambda - 1) = 0.$$

Thus $\lambda^2 - \lambda - 1 = 0$, and there are two such elements of $\mathcal{F}$. For the rest of the notes, let

$$\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618, \qquad \overline{\Phi} = \frac{1-\sqrt{5}}{2} \approx -.618.$$

These are the roots of the quadratic, $\Phi^2 = \Phi + 1$, $\overline{\Phi}^2 = \overline{\Phi} + 1$, and are related by $\Phi + \overline{\Phi} = 1$, $\Phi - \overline{\Phi} = \sqrt{5}$, $\Phi\overline{\Phi} = -1$. We now know that

$$p_\Phi = (1, \Phi, \Phi^2, \dots) \qquad p_{\overline{\Phi}} = (1, \overline{\Phi}, \overline{\Phi}^2, \dots)$$

are in $\mathcal{F}$. Since $\Phi \neq \overline{\Phi}$, p_Φ and $p_{\overline{\Phi}}$ are linearly independent, hence they span $\mathcal{F}$. This means that, in particular, there exist α and β so that $F = \alpha p_\Phi + \beta p_{\overline{\Phi}}$. Looking (as before) at the first two components, we have

$$0 = \alpha + \beta; \qquad 1 = \alpha\Phi + \beta\overline{\Phi}.$$

Hence $\beta = -\alpha$ and $\alpha = (\Phi - \overline{\Phi})^{-1} = 5^{-1/2}$. Putting this together, we find that

$$F_n = \frac{\Phi^n - \overline{\Phi}^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

This is called the *Binet formula* after Jacques Philippe Marie Binet (1786-1856), who discoverd it in 1843. (Binet was studying the Euclidean algorithm in what is apparently the first computational analysis in any algorithm.) Whevever you get an explicit formula for a sequence that is defined recursively, look at the asymptotics, that is, the approximate behavior for large n. Since $|\overline{\Phi}| < 1$, we have $\overline{\Phi}^n \to 0$, and in fact F_n is the *closest* integer to $\frac{1}{\sqrt{5}}\Phi^n$. This easily implies that $F_{n+1}/F_n \to \Phi$, and more to the point, that

$$\lim_{n\to\infty}\left(\frac{F_{n+2}}{F_{n+1}} - \frac{F_{n+1}}{F_n}\right) = 0.$$

Play with the numerator here! A representative example is $\frac{F_{10}}{F_9} - \frac{F_9}{F_8} = \frac{55}{34} - \frac{34}{21} = -\frac{1}{714}$.

Another peculiarity of the Binet formula is that it doesn't look like it gives even a rational number for F_n ... but remember the Binomial Theorem. We have

$$(a+b)^n = \sum_k \binom{n}{k} a^{n-k} b^k, \qquad (a-b)^n = \sum_k \binom{n}{k} (-1)^k a^{n-k} b^k;$$

since $\binom{n}{k} = 0$ if $k < 0$ or $k > n$, there is no need to be more specific about the limits on the sum. By adding and subtracting, noting that the terms $1 \pm (-1)^k$ take the values 0 and 2, depending on the parity of k mod 2, and then writing only the terms we need, we have

$$(a+b)^n + (a-b)^n = 2\sum_j \binom{n}{2j} a^{n-2j} b^{2j};$$
$$(a+b)^n - (a-b)^n = 2\sum_j \binom{n}{2j+1} a^{n-2j-1} b^{2j+1}.$$

Putting in $a = 1$ and $b = \sqrt{5}$, we obtain another formula:

$$F_n = \frac{2}{2^n\sqrt{5}} \sum_j \binom{n}{2j+1} (\sqrt{5})^{2j+1} = \frac{1}{2^{n-1}} \sum_j \binom{n}{2j+1} 5^j.$$

This at least is rational, even if it's not obvious that the sum will have a large power of 2 dividing it. As a check, if $n = 7$, then

$$\frac{1}{64}\left(\binom{7}{1} + \binom{7}{3}\cdot 5 + \binom{7}{5}\cdot 5^2 + \binom{7}{7}\cdot 5^3\right) =$$
$$\frac{1}{64}(7 + 35\cdot 5 + 21\cdot 25 + 1\cdot 125) = \frac{7+175+525+125}{64} = \frac{832}{64} = 13 = F_7.$$

The Binet formula gives us considerable insight into $\frac{F_{kn}}{F_n}$. We have

$$\frac{F_{2n}}{F_n} = \frac{\dfrac{\Phi^{2n} - \overline{\Phi}^{2n}}{\sqrt{5}}}{\dfrac{\Phi^n - \overline{\Phi}^n}{\sqrt{5}}} = \Phi^n + \overline{\Phi}^n.$$

Thus, we see directly that $\frac{F_{2n}}{F_n} \in \mathcal{F}$. This sequence is traditionally called L_n, after François Edouard Anatole Lucas (1842-1891). It's easy to compute $L_1 = F_2/F_1 = 1$, $L_2 = F_4/F_2 = 3$, $L_3 = F_6/F_3 = 4$, and by comparing with our earlier result, we see that $L_n = F_{n+1}+F_{n-1}$. (This can be verified from Binet's formula directly, as can most of the formulas we derive; I'll show this one in detail.)

$$L_n = F_{n+1} + F_{n-1} = \frac{1}{\sqrt{5}}\left(\Phi^{n+1} + \Phi^{n-1} - \overline{\Phi}^{n+1} - \overline{\Phi}^{n-1}\right)$$
$$= \left(\frac{\Phi + \Phi^{-1}}{\sqrt{5}}\right)\Phi^n - \left(\frac{\overline{\Phi} + \overline{\Phi}^{-1}}{\sqrt{5}}\right)\overline{\Phi}^n.$$

Since $\Phi + \Phi^{-1} = \Phi - \overline{\Phi} = \sqrt{5}$ and $\overline{\Phi} + \overline{\Phi}^{-1} = \overline{\Phi} - \Phi = -\sqrt{5}$, the formula is as before. We now have two reasons to say that $L_0 = 2$, from the formula, and from considering $L_2 - L_1$. We also have

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{1}{2^{n-1}} \sum_j \binom{n}{2j} 5^j.$$

We can continue the formulas:

$$\frac{F_{3n}}{F_n} = \frac{\frac{\Phi^{3n} - \overline{\Phi}^{3n}}{\sqrt{5}}}{\frac{\Phi^n - \overline{\Phi}^n}{\sqrt{5}}} = \Phi^{2n} + \Phi^n\overline{\Phi}^n + \overline{\Phi}^n = L_{2n} + (-1)^n.$$

Combining with our previous formula, we get

$$\frac{F_{3n}}{F_n} = L_{2n} + (-1)^n = \frac{F_{4n}}{F_{2n}} + (-1)^n.$$

If you write $M_n = \frac{F_{3n}}{F_n}$, then the first few terms are

$$M_1 = 2, \quad M_2 = 8, \quad M_3 = 17, \quad M_4 = 48, \quad M_5 = 122, \quad M_6 = 323, \ldots.$$

It's obvious that M_n is not a sequence in $\mathcal{F}$. Can you find some other recurrence satisfied by M_n? Note also that M_1 divides M_2, M_4, M_5, but not M_3, M_6 and M_2 divides M_4 but not M_6 and M_3 divides M_6. Seek patterns! Note that we also have, on cross-multiplying,

$$F_{3n}F_{2n} = F_nF_{4n} + (-1)^nF_n.$$

It is certainly true asymptotically, that if a, b, c, d are large and $a + b = c + d$, then $F_aF_b \approx F_cF_d$. You can try to parameterize this by assuming that a, b, c, d are all positive and different, and, without loss of generality, a is the smallest. Then $b = a + j, c = a + k$ for some positive j and k. This forces $d = a + j + k$. What is $F_{a+j+k}F_a - F_{a+j}F_{a+k}$?

You could also play with $\frac{F_{kn}}{F_n}$ for $k = 4, 5$, and see what additional formulas you can find. The algebraic reason that the formulas are so simple is that $x - y$ is always a factor of $x^k - y^k$. You can also try to find formulas for $\frac{L_{kn}}{L_n}$, Since $x + y$ is a factor of $x^k + y^k$ only when k is odd, these quotients are more interesting if k is odd.

The asymptotics of L_n suggest that $L_n \approx \Phi^n \approx \sqrt{5}F_n$. Thus, not only can you look at $\frac{L_{n+2}}{L_{n+1}} - \frac{L_{n+1}}{L_n}$ but $L_n - \sqrt{5}F_n$. To make this an integer, rationalize and consider $L_n^2 - 5F_n^2$.

Thinking bigger, any element of $x \in \mathcal{F}$ has a formula $x_n = \alpha\Phi^n + \beta\overline{\Phi}^n$, so if $\alpha \neq 0$ (that is, unless x is a multiple of $P_{\overline{\Phi}}$), $x_n \approx \alpha\Phi^n$ for large n, so you might want to look at $\frac{x_{n+2}}{x_{n+1}} - \frac{x_{n+1}}{x_n}$. You might also want to look at the asymptotics of $x_{n+1}/x_n - \Phi$, or more to the point, $(x_{n+1}/x_n - \Phi)^{-1}$.

There's another way that we can use linear algebra to find the Binet formula. We change our focus from F_n to the *pair* (F_n, F_{n+1}). To be precise,

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}.$$

This equation easily iterates to:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The standard techniques of linear algebra will let us write $A = XDX^{-1}$ where D is a diagonal matrix, and $A^n = (XDX^{-1})(XDX^{-1})\cdots(XDX^{-1}) = XD^nX^{-1}$ will allow us to compute F_n, in a different way than before.

Let's suppose you don't remember exactly how to do this. You play around and remember that $A = XDX^{-1} \iff AX = XD$, and you also remember that eigenvectors somehow show up. The characteristic polynomial of A is $\left|\begin{smallmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{smallmatrix}\right|$ is a familiar quadratic: $(-\lambda)(1-\lambda) - 1 = \lambda^2 - \lambda - 1$. The roots of the characteristic polynomial are our old friends Φ and $\overline{\Phi}$. Thus these are the eigenvalues for A, and it's easy to find the eigenvectors:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \Phi - 1 \end{bmatrix} = \begin{bmatrix} \Phi \\ 1 \end{bmatrix} = \Phi \begin{bmatrix} 1 \\ \Phi - 1 \end{bmatrix},$$

since $1 = \Phi(\Phi - 1)$ from the characteristic equation. The same thing holds for $\overline{\Phi}$, and so

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \Phi - 1 & \overline{\Phi} - 1 \end{bmatrix} = \begin{bmatrix} \Phi & \overline{\Phi} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \Phi - 1 & \overline{\Phi} - 1 \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \overline{\Phi} \end{bmatrix}.$$

That is, $X = \left[\begin{smallmatrix} 1 & 1 \\ \Phi-1 & \overline{\Phi}-1 \end{smallmatrix}\right]$ and $D = \left[\begin{smallmatrix} \Phi & 0 \\ 0 & \overline{\Phi} \end{smallmatrix}\right]$. Thus,

$$(*) \qquad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \Phi - 1 & \overline{\Phi} - 1 \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \overline{\Phi} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \Phi - 1 & \overline{\Phi} - 1 \end{bmatrix}^{-1}.$$

Since $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$ when defined, (*) becomes

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \Phi - 1 & \overline{\Phi} - 1 \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \overline{\Phi} \end{bmatrix} \begin{bmatrix} \overline{\Phi} - 1 & -1 \\ 1 - \Phi & 1 \end{bmatrix} \frac{1}{(\overline{\Phi} - 1) - (\Phi - 1)}.$$

Since $\overline{\Phi} - \Phi = -\sqrt{5}$, and $\Phi + \overline{\Phi} = 1$, so that $\overline{\Phi} - 1 = -\Phi, \Phi - 1 = -\overline{\Phi}$, we have

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n &= \begin{bmatrix} 1 & 1 \\ -\overline{\Phi} & -\Phi \end{bmatrix} \begin{bmatrix} \Phi^n & 0 \\ 0 & \overline{\Phi}^n \end{bmatrix} \begin{bmatrix} -\Phi & -1 \\ \overline{\Phi} & 1 \end{bmatrix} \frac{1}{-\sqrt{5}} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \Phi^{n+1} - \overline{\Phi}^{n+1} & \Phi^n - \overline{\Phi}^n \\ \Phi\overline{\Phi}^{n+1} - \overline{\Phi}\Phi^{n+1} & \Phi\overline{\Phi}^n - \overline{\Phi}\Phi^n \end{bmatrix} \end{aligned}$$

Since $\Phi\overline{\Phi} = -1$, this reduces to

$$A^n = \frac{1}{\sqrt{5}} \begin{bmatrix} \Phi^{n+1} - \overline{\Phi}^{n+1} & \Phi^n - \overline{\Phi}^n \\ \Phi^n - \overline{\Phi}^n & \Phi^{n-1} - \overline{\Phi}^{n-1} \end{bmatrix}.$$

hence, by plugging back into the original equation, we obtain

$$\begin{aligned} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \frac{1}{\sqrt{5}} \begin{bmatrix} \Phi^{n+1} - \overline{\Phi}^{n+1} & \Phi^n - \overline{\Phi}^n \\ \Phi^n - \overline{\Phi}^n & \Phi^{n-1} - \overline{\Phi}^{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \Phi^{n+1} - \overline{\Phi}^{n+1} \\ \Phi^n - \overline{\Phi}^n \end{bmatrix} \end{aligned}$$

This gives the Binet formula, and furthermore,

$$A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

Since matrix multiplication is associative (as we used repeatedly above), the equation $A^{m+n} = A^m A^n$ implies that

$$\begin{bmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{bmatrix} = \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{n-1} \end{bmatrix} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

Thus,

$$\begin{aligned} F_{m+n+1} &= F_{m+1}F_{n+1} + F_m F_n; \\ F_{m+n} &= F_{m+1}F_n + F_m F_{n-1} = F_m F_{n+1} + F_{m-1}F_n; \\ F_{m+n-1} &= F_m F_n + F_{m-1}F_{n-1}. \end{aligned}$$

as we have already seen. Since the Lucas numbers satisfy the Fibonacci recurrence, but with the initial conditions $L_0 = 2, L_1 = 1$, we have

$$\begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

You may want to play around with the arguments mentioned above in order to rederive the closed formula for L_n. If you like linear algebra, you might want to view A as representing the shift operator on the vector space $\mathcal{F}$.

One final question. We have expressed F_{n+k} in terms of F_n and F_{n+1}. Could we have used F_n and F_{n+2}? Sure. In fact, we could have used F_n and F_{n+j} for any $j \geq 1$. Let's fix j and consider the elements F and $F^{'\cdots'} = (F_j, F_{j+1}, \ldots) \in \mathcal{F}$. Since $F = (0, 1, \ldots)$, these two elements are linearly independent if $F_j \neq 0$, which will certainly be the case here, so

$$(F_k, F_{k+1}, F_{k+2}, \ldots) = \alpha(F_0, F_1, F_2, \ldots) + \beta(F_j, F_{j+1}, F_{j+2} \ldots)$$

if and only if they agree on the first two terms; that is,

$$F_k = \alpha \cdot 0 + \beta \cdot F_j \qquad F_{k+1} = \alpha \cdot 1 + \beta \cdot F_{j+1} \implies \alpha = F_{k+1} - \frac{F_k}{F_j}, \quad \beta = \frac{F_k}{F_j}.$$

You can't expect the formula for β to simplify much, but the formula for α becomes

$$\alpha = \frac{F_j F_{k+1} - F_{j+1}F_k}{F_j}.$$

The tedious, guaranteed way to understand the numerator is to expand by Binet's formula. It's more fun to use the recurrence to give a clue:

$$F_j F_{k+1} - F_{j+1}F_k = F_j(F_k + F_{k-1}) - (F_j + F_{j-1})F_k = -(F_{j-1}F_{(k-1)+1} - F_{(j-1)+1}F_k).$$

There's an induction here, and you will need to split the cases $j > k$ and $j < k$.