Notes on the Fibonacci numbers, Math 296, 1/22/01

These are intended to be some introductory notes on the Fibonacci numbers, illustrating the many different ways of thinking about them.

The sequence of integers $\{F_n\}$ is defined recursively by:

$$(*) \qquad F_0 = 0,\ F_1 = 1; \qquad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2$$

The first few elements of the sequence are:

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21,$$
$$F_9 = 34, F_{10} = 55, F_{11} = 89, F_{12} = 144, F_{13} = 233, F_{14} = 377, F_{15} = 610.$$

As one approach, we decide to look at the totality of sequences which satisfy the reccurence of the Fibonacci sequences. (One of the principles for research being to embed examples in more general frameworks.) This one is motivated by the observation that the initial conditions $(F_0, F_1) = (0, 1)$ might be generalized.

Let $\mathcal{F}$ denote the set of all sequences

$$x = (x_0, x_1, x_2, \ldots), \quad x_n \in \mathbf{C}.$$

with the property that $x_n = x_{n-1} + x_{n-2}$ for $n \geq 2$. (There's actually no reason to restrict the elements of the sequence to be complex numbers, but there's no reason at the moment not to.) Two distinguished elements of $\mathcal{F}$ are the Fibonacci sequence F and what I'll call F' for the shifted Fibonacci sequence:

$$F = (0, 1, 1, 2, 3, 5, \ldots) \qquad F' = (1, 1, 2, 3, 5, 8, \ldots).$$

That is, the n-th element of F is just F_n, and $(F')_n = F_{n+1}$.

We can make $\mathcal{F}$ into a vector space over $\mathbf{C}$ by defining addition and scalar multiplication in the "obvious" way: if $c \in \mathbf{C}$ and x and $y = (y_0, y_1, y_2, \ldots)$ are in $\mathcal{F}$, then

$$cx = (cx_0, cx_1, cx_2, \ldots), \qquad x + y = (x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots).$$

It's easy to check that cx and $x + y$ are both in $\mathcal{F}$ by verifying the recurrence, and The zero element in $\mathcal{X}$, 0, is simply the sequence $(0, 0, 0, \ldots)$.

If we write out the recurrence, we see that if $x \in \mathcal{F}$, then $x_2 = x_0 + x_1$, $x_3 = x_0 + 2x_1$, $x_4 = 2x_0 + 3x_1 \ldots$ and so on. In fact, it is easy to show by induction that there exist numbers a_n and b_n, *independent* of x_0 and x_1 so that $x_n = a_n x_0 + b_n x_1$. Furthermore, the sequences a and b satisfy the Fibonacci recurrence and so are elements of $\mathcal{F}$, Indeed,

$$x = (x_0, x_1, x_0 + x_1, x_0 + 2x_1, 2x_0 + 3x_1, \ldots) =$$
$$x_0(1, 0, 1, 1, 2, \ldots) + x_1(0, 1, 1, 2, 3, \ldots)$$

This means that $\mathcal{F}$ is at most a two dimensional subspace of $\mathcal{F}$, and since the two elements of the spanning set are obviously linearly independent, $\mathcal{F}$ is two dimensional, and they form a basis. The second element we recognize as F. The first one looks like F shifted the other way, so we want to say that the n-th element is F_{n-1}, but that would require defining F_{-1} as the first element. An alternative is to note that $F_{n-1} = F_{n+1} - F_n$ from the recurrence, so the sequence is $F' - F$. In sum, we've shown that if $x \in \mathcal{F}$, then

$$x = x_0(F' - F) + x_1 F \implies x_n = x_0(F_{n+1} - F_n) + x_1 F_n = (-x_0 + x_1)F_n + x_0 F_{n+1}.$$

We are now allowed to harvest some results from this. Consider the sequence $x^{(k)} = F^{''\cdots'}$, where there are k primes; that is, $x_n = F_{n+k}$, with $k \geq 0$. This is F shifted k times, and it's clear that $x^{(k)} \in \mathcal{F}$, with $x_0^{(k)} = F_k$ and $x_1^{(k)} = F_{k+1}$. Then the formula above implies that

$$x_n^{(k)} = F_{n+k} = F_k(F_{n+1} - F_n) + F_{k+1}F_n.$$

There are two ways to rewrite this: one is symmetric and one is short.

$$F_{n+k} = F_k F_{n+1} + F_{k+1} F_n - F_k F_n;$$
$$F_{n+k} = F_k F_{n-1} + F_{k+1} F_n.$$

The understanding here is that $n \geq 1$ so that we don't have a negative index. Because the second formula is symmetric in k and n on the left, but not the right, we get a second formula,

$$F_{n+k} = F_n F_{k-1} + F_{n+1} F_k,$$

and an identity which isn't too hard to derive directly:

$$F_k F_{n-1} + F_{k+1} F_n = F_n F_{k-1} + F_{n+1} F_k,$$

Here are three questions I won't answer, but you might want to:

(i) What is a "good" formula for $F_{n_1+n_2+n_3}$? There are two metrics for "good". One is that you want a symmetric formula, the other is that you want a short formula. (The best way to proceed is to use the previous work and write $n_1 + n_2 + n_3 = (n_1 + n_2) + n_3$.)

(ii) Look for a converse. Suppose f_n is a sequence and, for all $k, n \geq 0$, we have $f_{n+k} = f_k f_{n+1} + f_{k+1} f_n - f_k f_n$. Does this mean that $f_n = F_n$ for all n?

(iii) To what extent is the abstract structure dependent on the specific linear recurrence? What would happen if you looked at all sequences with the property $x_n = \alpha x_{n-1} + \beta x_{n-2}$ for fixed constants α and β. Would you still get a two-dimensional vector space? What do you think would happen if you considered the property $x_n = \alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}$? What about $x_n = x_{n-1}^2 + x_{n-2}^2$?

Some questions I will discuss are: what happens when you "diagonalize" the symmetry by setting the parameters n and k equal to each other or nearly so, and whether there are other natural bases for $\mathcal{F}$.