1. §2.5 – 27 aceh. In each case, the singularity of \( f \) at infinity is the same as the singularity of \( g(z) = f(1/z) \) at \( z = 0 \), with the understanding that \( g(0) \) is not defined so that it’s always an isolated singularity. First \( f(z) = 3z^2 + 4 - \frac{1}{z} \implies g(z) = \frac{3}{z^2} + 4 - z \). This gives a pole of order two at \( z = 0 \), and that’s what \( f \) has at \( \infty \). And \( f \), as given, is already a Laurent series. Second,

\[
f(z) = \frac{z^2}{z - 4} \implies g(z) = \frac{1}{\frac{1}{z^2} - 4} = \frac{1}{z(1 - 4z)} = \frac{1}{z} + 4 + 16z + \ldots.
\]

so \( f \) has a pole of order one at \( \infty \) and its Laurent series is \( z + 4 + \frac{16}{z} + \ldots \). Third, \( f(z) = e^{1/z} \implies g(z) = e^z \), so \( f \) has a removable singularity at \( \infty \). We have \( \lim_{z \to \infty} e^{1/z} = e^0 = 1 \), and the usual series: \( e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots \). Finally, \( f(z) = \sin \frac{1}{z} \implies g(z) = \sin z \), so \( f \) has a removable singularity at \( \infty \) with \( \lim_{z \to \infty} \sin \frac{1}{z} = \sin 0 = 0 \) and Laurent series \( f(z) = \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120z^5} + \ldots \).

2. §2.5 – 22 e. With \( e^z \), \( \sin z \) and \( \cos z \), we have power series that converge for all \( z \), so we can use them with abandon at every opportunity. In particular, for all \( z \neq 0 \),

\[
z \cos \frac{1}{z} = z \left( 1 - \frac{1}{2z^2} + \frac{1}{24z^4} - \ldots \right) = z - \frac{1}{2z} + \frac{1}{24z^3} - + \ldots.
\]

3. §2.6 – 6. We take the usual protractor contour to \( f(z) = \frac{e^{iz}}{z^2 + 6z + 10} \), which has simple poles at \( z = -3 \pm i \) and obtain:

\[
\int_{-R}^{R} \frac{e^{ix}}{x^2 + 6x + 10} \, dx + \int \frac{e^{iz}}{z^2 + 6z + 10} \, dz = 2\pi i \cdot \text{Res}(f, -3 + i).
\]

Some calculations are now necessary. In the upper half plane, \( |e^{iz}| \leq 1 \), so that

\[
\left| \int \frac{e^{iz}}{z^2 + 6z + 10} \, dz \right| \leq \frac{\pi R}{R^2 - 6R - 10} \to 0
\]
as \( R \to \infty \); we don’t need Jordan’s lemma here. Further,

\[
\text{Res}(f, -3 + i) = \frac{e^{i(-3+i)}}{2((-3 + i) + 6} = \frac{\cos 3 - i \sin 3}{2ei}.
\]

Taking the limit as \( R \to \infty \), we conclude that

\[
\int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x^2 + 6x + 10} \, dx = 2\pi i \left( \frac{\cos 3 - i \sin 3}{2ei} \right) = \frac{\pi \cos 3}{e} - i \frac{\pi \sin 3}{e}.
\]
Taking the imaginary parts above, we see that

\[
\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 6x + 10} \, dx = -\frac{\pi \sin 3}{e}.
\]

Another way of doing this problem is to make a change of variables at the very beginning of \(x \to x - 3\) and then use trig identities to get

\[
\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 6x + 10} \, dx = \int_{-\infty}^{\infty} \frac{\sin(x - 3)}{x^2 + 1} \, dx
\]

\[
= (\cos 3) \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 1} \, dx - (\sin 3) \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} \, dx.
\]

The values of these integrals are found in Example 3.

4. §2.6 – 12. Here, we use \(\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\), and make the mechanical substitution:

\[
\int_{\theta = 0}^{2\pi} \sin^{2k} \theta \, d\theta = \int \left( \frac{z - 1/z}{2i} \right)^{2k} \frac{dz}{iz} = \frac{1}{(2i)^{2k}i} \int_{|z| = 1} \left( \frac{z - 1}{z} \right)^{2k} \frac{dz}{z}.
\]

This isn’t as bad as it looks, because all we need to do is find the coefficient of \(1/z\) in the integrand, in order to determine the residue. Because of the factor of \(1/z\), this is the constant term in \((z - 1/z)^{2k}\), which is \((-1)^k (2k\choose k)\) by the binormal theorem. Since \(i^{2k} = (-1)^k\), the value of the integral is therefore

\[
2\pi i \cdot \frac{(-1)^k (2k\choose k)}{(2i)^{2k}i} = \frac{\pi (2k\choose k)}{2^{2k-1}} = \frac{\pi (2k)!}{2^{2k-1}(k!)^2}.
\]

5 and 6. (E) Compute, with an explicit discussion of the behavior on the semicircle, \(I = \int_0^\infty \frac{x^2}{(x^2 + 9)^3} \, dx\). Note the limits of integration and the fact that the integrand is even. Since the integrand is even, \(I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 9)^3} \, dx\). To evaluate \(I\), we let \(f(z) = \frac{x^2}{(x^2 + 9)^3}\), which has a poles of order 3 at \(\pm 3i\), and use the protractor contour to find that

\[
\int_{-R}^{R} \frac{x^2}{(x^2 + 9)^3} \, dx + \int_{\gamma} \frac{z^2}{(z^2 + 9)^3} \, dz = 2\pi i \cdot \text{Res}(f(z), 3i).
\]

This problem asked for a specific discussion of the second term. The path has length \(\pi R\), and when \(z = Re^{it}\), \(|z^2| = R^2\), while \(|z^2 + 9| \geq R^2 - 9\). Thus,

\[
\left| \int_{\gamma} \frac{z^2}{(z^2 + 9)^3} \, dz \right| \leq \pi R \frac{R^2}{(R^2 - 9)^3} \to 0
\]
as \( R \to \infty \). Furthermore, \( f(z) = H(z)/(z-3i)^3 \), where \( H(z) = z^2(z+3i)^{-3} \), so the residue is equal to \( \frac{1}{2!}H''(3i) \). We have

\[
H'(z) = 2z(z+3i)^{-3} + z^2(-3)(z+3i)^{-4} \implies H''(z) = 2(z+3i)^{-3} + 2z(-3)(z+3i)^{-4} + 2z(-3)(z+3i)^{-4} + z^2(-3)(-4)(z+3i)^{-5}
\]

\[
\implies H''(3i) = \frac{2}{(6i)^3} - \frac{12}{(6i)^4} + \frac{12(3i)^2}{(6i)^5} = -\frac{i}{216}
\]

\[
\implies \int_{0}^{\infty} \frac{x^2}{(x^2 + 9)^3} \, dx = 2\pi i \cdot \frac{-i}{216} = \frac{\pi}{216}.
\]

7. (\( E \)) Observe that \( z^4 + 4 = (z^2 - 2z + 2)(z^2 + 2z + 2) \), and use this fact to evaluate \( \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 4} \, dx \). This is an instance where the book’s theorem is applicable, since the degree of the denominator is 4 and the degree of the numerator is 2. Since the integrand has simple poles at \(-1 \pm i\) and \(1 \pm i\), the integral equals

\[
2\pi i \left[ \text{Res} \left( \frac{x^2}{x^4 + 4}; -1 + i \right) + \text{Res} \left( \frac{x^2}{x^4 + 4}; 1 + i \right) \right] = 2\pi i \left( \frac{(-1 + i)^2}{4(-1 + i)^3} + \frac{(1 + i)^2}{4(1 + i)^3} \right)
\]

\[
= \frac{2\pi i}{4} \left( \frac{1}{-1 + i} + \frac{1}{1 + i} \right) = \frac{\pi i}{2} \left( \frac{-1 - i}{2} + \frac{1 - i}{2} \right) = \frac{\pi}{2}.
\]

8. Let \( f(z) = \frac{1}{z} \left( \frac{1}{z} + \frac{1}{z-1} \right) \). Find, carefully, Laurent series for \( f \) which converge in each of the following regions: (a) \( |z - 2| < 1 \), (b) \( 1 < |z - 2| < 2 \), (c) \( 2 < |z - 2| \). Let’s take the two separately, and to save time, I won’t put in all the intermediate steps – we’ve done them several times already.

\[
\frac{1}{z} = \frac{1}{2 + (z-2)} = \frac{1/2}{1 + (z-2)/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^n \quad \text{if } |z-2| < 2,
\]

\[
\frac{1}{z} = \frac{1}{2 + (z-2)} = \frac{1/(z-2)}{1 + 2/(z-2)} = \sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{(z-2)^n} \quad \text{if } |z-2| > 2;
\]

\[
\frac{1}{z-1} = \frac{1}{1 + (z-2)} = \sum_{n=0}^{\infty} (-1)^n (z-2)^n \quad \text{if } |z-2| < 1,
\]

\[
\frac{1}{z-1} = \frac{1}{1 + (z-2)} = \frac{1/(z-2)}{1 + 1/(z-2)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(z-2)^n} \quad \text{if } |z-2| > 1.
\]

Putting the right series together for the appropriate cases, the answers are:

(a) \( \sum_{n=0}^{\infty} (-1)^n \left( 1 + \frac{1}{2^{n+1}} \right) (z-2)^n; \)

(b) \( \sum_{n=0}^{\infty} \left( \frac{(-1)^n}{2^{n+1}} \right) (z-2)^n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(z-2)^n}; \)

(c) \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1 + 2^{n-1})}{(z-2)^n}. \)
9. (E) Classify the singularities (in the complex plane plus infinity) and including the order of the poles if relevant, of the following functions: \( f_1(z) = \frac{(z+2)^2}{z^3(z-1)}, \ f_2(z) = \frac{(e^z-1)^2}{z^4}. \) Well, for \( f_1, \) clearly there is a pole of order 3 at \( z = 0 \) and a pole of order 1 at \( z = 1. \) To check at \( \infty, \) a calculation shows that \( f_1(1/z) = \frac{z^2(1+2z)^2}{1-z}, \) which has a zero of order two at \( z = 0, \) hence \( f_1 \) has a removable singularity at \( \infty. \) The only possible finite singularity for \( f_2 \) is at \( z = 0; \) since \( e^z - 1 \) has a zero of order 1, the numerator has a zero of order two and the denominator has a zero of order 4, hence \( f_2 \) has a pole of order two at \( 0. \) To check at \( \infty, \) we look at \( f_2(1/z) = z^4(e^{1/z} - 1)^2. \) As \( z \to 0, \) the high weirdness of the numerator implies that we have an essential singularity; hence \( f_2 \) has an essential singularity at \( \infty. \) Less intuitively, we could look at the Laurent series for \( f_2(1/z), \) convergent in \( 0 < |z| < \infty. \) Since there are clearly terms of arbitrarily negative exponent, it’s an essential singularity:

\[
z^4(e^{1/z} - 1)^2 = z^4 \left( \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \ldots \right)^2
\]

10. (E) Let \( f(z) = \frac{e^{iz}}{z^2 + 1}. \)

a. Find a constant \( M_1 \) so that \( |f(z)| \leq M_1 \) for \( z \) on the real interval \([-10, 10]\).

Well \( |e^{iz}| = 1, \) so \( |f(x)| = \frac{1}{1 + x^2}, \) and since the denominator is \( \geq 1, \) we can take \( M_1 = 1. \)

b. Find a constant \( M_2 \) so that \( |f(z)| \leq M_2 \) for \( z \) on the upper semicircle \( z = 10 \text{e}^{it}, \) \( 0 \leq t \leq \pi. \)

We’ve seen that \( |e^{i(x+iy)}| = e^{-y}, \) so if \( y \geq 0, \) then the numerator is bounded by 1, and the denominator is at least \( 10^2 - 1, \) so \( M_2 = \frac{1}{99} \) is defensible.

c. Find a constant \( M_3 \) so that \( |f(z)| \leq M_3 \) for \( z \) on the lower semicircle \( z = 10 \text{e}^{it}, \) \( \pi \leq t \leq 2\pi. \)

Alas, nothing good can be said about the numerator: \( e^{-y} \) can be as large as \( e^{10}, \) the denominator information is the same, and the best we can say is that \( M_3 = \frac{e^{10}}{99}. \)

Some people might be interested in the actual best upper bounds. Observe that \( f(0) = 1, \) so \( M_1 \) is best possible, and \( f(-10i) = \frac{e^{10}}{99}, \) so \( M_3 \) is best possible. It turns out that \( M_2 \) is not best possible. Let’s get into the details. Write \( z = x + iy, \) where \( y \geq 0 \) and \( x^2 + y^2 = 10^2. \) Then

\[
|f(z)|^2 = \left| \frac{e^{ix-y}}{(x^2 - y^2 + 1) + i(2xy)} \right|^2 = \frac{e^{-2y}}{(x^2 - y^2 + 1)^2 + (2xy)^2} = \frac{e^{-2y}}{101^2 - 4y^2}.
\]

Let \( \phi(y) = \frac{e^{-2y}}{101^2 - 4y^2}. \) Then it’s easy to check that \( \phi'(y) < 0 \) for \( 0 \leq y \leq 1, \) so that the maximum of \( |f(z)|^2 \) occurs when \( y \) is minimized, i.e., when \( z = \pm 10 \) and \( y = 0. \) Thus, \( M_2 = \frac{1}{101} \) is actually a correct answer, but it requires more work than use of the incorrect inequality \( \left| \frac{1}{z^2 + 1} \right| \leq \left| \frac{1}{|z|^2 + 1}. \right. \)