1. §2.4 – 10. “Find the power-series expansion about the given point for each of the functions; find the largest disc in which the series is valid”: here, $e^z$ about $z_0 = \pi i$. If $f(z) = e^z$, then $f^{(n)}(z) = e^z$ for all $n \in \mathbb{N}$. Since the fundamental formula is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

and $e^{\pi i} = -1$, we obtain the odd-looking formula:

$$e^z = \sum_{n=0}^{\infty} \frac{-1}{n!}(z - \pi i)^n.$$

(This isn’t so strange if you think about it: the right hand side is simply $-e^{z-\pi i}$.) Since $f$ is entire, it is analytic on disks of the form $|z - \pi i| < R$ for every $R$. Thus, the power series converges on each such disk, and therefore converges for all $z$.

2. §2.4 – 12. Same question, for $\frac{z^2}{1-z}$ at $z_0 = 0$. There are a lot of ways to do this problem. The fastest is to multiply the respective power series for $z^2$ at $z_0 = 0$ and $\frac{1}{1-z}$ at $z_0 = 0$:

$$(z^2) \sum_{n=0}^{\infty} z^n = \sum_{n=2}^{\infty} z^n = z^2 + z^3 + \ldots$$

Another way of doing this is to divide the numerator into the denominator:

$$\frac{z^2}{1-z} = \frac{(z^2 - 1) + 1}{1-z} = -1 - z + \frac{1}{1-z}.$$

The first two terms of $\frac{1}{1-z}$ are erased by $-1 - z$ as before. The series converges precisely for $|z| < 1$, as we already know, and this is the largest possible disk.

3. §2.4 – 20. Suppose $f$ is entire and $Re(f(z)) < c$ for all $z$. Following the hint, and writing $g(z) = e^{f(z)}$, we note that $g$ is entire and $|g(z)| = e^{Re(f(z))} < e^c$. It then follows by Liouville’s Theorem that $g$ is constant, say $g(z) = k \neq 0$. A small argument is necessary now to show that $f$ is constant: we know that $e^{f(z)} = k$. Thus, for all $z$, $f(z) = \log(k) + 2\pi in(z)$ for some integer $n(z)$ which might depend on $z$. However, $f$ is continuous (since it is entire), and this means that $n(z)$ is continuous. And a continuous function which takes only integer values is constant, so $f$ is constant. (If you don’t know the proof of this last remark, think about the definition of continuity with $\epsilon = 1/3$.)

4. §2.4 – 27 and 28a (note that the solution to 27 is essentially given in the back.) Suppose $f$ and $A$ are analytic in a simply-connected domain $D$ and $f'(z) = A(z)f(z)$ for $z \in D$. You are asked to prove that

$$f(z) = Ce^{\int_{z_0}^z A(w) \, dw}$$
for some constant $C$, where the integral is taken over any piecewise smooth path connecting
a basepoint $z_0$ to $z$. Again following the hint, let
\[ g(z) = e^{- \int_{z_0}^{z} A(w) \, dw} . \]
Since $A$ is analytic in a simply-connected domain, the function $\int_{z_0}^{z} A(w) \, dw$ is well-defined
and analytic in $D$, and an antiderivative for $A(z)$. Thus, by the chain rule, $g'(z) = g(z)(-A(z))$, and so,
\[ (f(z)g(z))' = f(z)g'(z) + f'(z)g(z) = -A(z)f(z)g(z) + A(z)f(z)g(z) = 0. \]
This means that $f(z)g(z)$ is constant on $D$; using $g(z_0) = e^0$, we have
\[ f(z)g(z) = f(z_0)g(z_0) = f(z_0) \implies f(z) = \frac{f(z_0)}{g(z)} = f(z_0)e^{\int_{z_0}^{z} A(w) \, dw} , \]
as desired. With $A(z) = -2z$, we can take $D = \mathbb{C}$, and for simplicity, $z_0 = 0$. Thus,
\[ f(z) = f(0)e^{\int_{0}^{z} 2w \, dw} = f(0)e^{z^2} . \]
5. (E) Evaluate two integrals:
\[ \frac{1}{2\pi i} \int_{|z|=1} \left( z + \frac{2}{z} \right)^3 \, dz = \frac{1}{2\pi i} \int_{|z|=1} \left( z^3 + 6z + \frac{12}{z} + \frac{8}{z^3} \right) \, dz = 12 , \]
since the integral of $z^n$ is 0 if $n \neq -1$ and 1 if $n = 1$. And
\[ \frac{1}{2\pi i} \int_{|z|=2} \frac{dz}{z^2 - 3z} = \frac{1}{2\pi i} \int_{|z|=2} \frac{dz}{z - 0} = \frac{1}{0 - 3} = -\frac{1}{3} , \]
since 0 is inside $|z| = 2$ and $\frac{1}{z-3}$ is analytic there. (Or use partial fractions.)
6. (E) Evaluate the following integrals, where $C$ denotes the contour $|z| = 2$, taken in the
usual counterclockwise way:
\[ \frac{1}{2\pi i} \int_{C} \frac{\cos z}{z} \, dz = \cos 0 = 1 \]
by the usual procedure, since $\cos z$ is entire, 0 is inside $C$ and $z = z - 0$;
\[ \frac{1}{2\pi i} \int_{C} \frac{e^{3z}}{z^4} \, dz = \frac{1}{3!} (e^{3z})'' \bigg|_{z=0} = \frac{3^3 e^{3z}}{6} \bigg|_{z=0} = \frac{9}{2} \]
by Cauchy’s Theorem, since $e^{3z}$ is entire and $z^4 = (z - 0)^4$;
\[ \frac{1}{2\pi i} \int_{C} e^{3z} (z - 1)^4 \, dz = 0 , \]
because the integrand is entire.

7. Evaluate the following integrals, where $C$ denotes the contour $|z| = 1$, taken in the usual counterclockwise way:

$$
\frac{1}{2\pi i} \int_C \frac{z}{e^z} \, dz = 0,
$$

because the integrand is entire;

$$
\frac{1}{2\pi i} \int_C \frac{e^{3z}}{(z - 3)^4} \, dz = 0,
$$

because the integrand, $ze^{-z}$, is analytic in $|z| < 5/2$ (for example), which contains $C$ and its interior; finally, since $-\frac{3}{4}$ lies within $C$,

$$
\frac{1}{2\pi i} \int_C \frac{1}{3 + 4z} \, dz = \frac{1}{4} \left( \frac{1}{2\pi i} \int_C z - (-\frac{3}{4}) \, dz \right) = \frac{1}{4}.
$$

8. (E) Let

$$f(z) = \frac{1}{1 - 2z} + \frac{1}{1 + z}.$$  

Write down the power series for $f$ centered at $z_0 = 0$ and and $z_0 = 3$. This can be done in at least two different ways: either by computing $f^{(n)}(z)$ by an easy induction and evaluation at $z_0$, or by manipulating the geometric series.

a. We have already seen that if $g(z) = (1 - z)^{-1}$, then $g^{(n)}(z) = n!(1 - z)^{-(n+1)}$. It follows from the chain rule that if $h(z) = (1 - az)^{-1}$, then $h^{(n)}(z) = n!a^n(1 - az)^{-(n+1)}$. Thus,

$$f^{(n)}(z) = n! \left( \frac{2^n}{(1 - 2z)^{n+1}} + \frac{(-1)^n}{(1 + z)^{n+1}} \right).$$

It follows that for $f^{(n)}(0) = n!(2^n + (-1)^n)$, and the power series at 0 is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (z - 0)^n = \sum_{n=0}^{\infty} (2^n + (-1)^n)z^n.$$  

This can also be found by adding the two basic geometric series.

b. Similarly,

$$f^{(n)}(3) = n! \left( \frac{2^n}{(-5)^{n+1}} + \frac{(-1)^n}{4^{n+1}} \right),$$

so that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} (z - 3)^n = \sum_{n=0}^{\infty} \left( -\frac{1}{5} \left( -\frac{2}{5} \right)^n 2^n + \frac{1}{4} \left( -\frac{1}{4} \right)^n \right) (z - 3)^n.$$
The way to find this without differentiating is to rewrite $f$ as a geometric series in $z - 3$:

$$f(z) = \frac{1}{1 - 2(z - 3) + 1} + \frac{1}{1 - 2(z - 3) + 3} = \frac{1}{-5 - 2(z - 3)} + \frac{1}{4 + z - 3}
= -\frac{1}{5} \cdot \frac{1}{1 + \frac{2}{5}(z - 3)} + \frac{1}{4} \cdot \frac{1}{1 + \frac{1}{4}(z - 3)}.$$ 

9a. (E) Suppose $C$ is a piecewise smooth, (not necessarily closed!) contour. Prove that $\int_C z \, dz = 0$ implies $\int_C z^3 \, dz = 0$. Let $C$ have beginning point $z_0$ and endpoint $z_1$. Then we've known for some time that

$$\int_C z^n \, dz = \frac{1}{n + 1} (z_1^{n+1} - z_0^{n+1}).$$

If $\int_C z \, dz = 0$, then $z_1^2 - z_0^2 = 0$, so $z_0^2 = z_1^2$, hence $z_0^3 = z_1^3$, so that $\int_C z^3 \, dz = 0$.

9b. Find a simple, piecewise smooth contour $C$ so that $\int_C z^3 \, dz = 0$ and $\int_C z \, dz = 1$. That would be a contour from $z_0$ to $z_1$ so that

$$0 = \frac{1}{4} (z_1^4 - z_0^4), \quad 1 = \frac{1}{2} (z_1^2 - z_0^2).$$

That is, $z_1^4 = z_0^4$ and $z_1^2 - z_0^2 = 2$. Since $z_1^4 = z_0^4$, we have $z_1^2 = \pm z_0^2$, and, when combined with the second equation, we must have $z_1^2 = 1$, $z_0^2 = -1$. Therefore, any contour which begins at $i$ or $-i$ and ends at $1$ or $-1$ will work.

10. Let $C$ denote a contour consisting of a line segment from $1 - 2i$ to $4i$ followed by a line segment from $4i$ to $2 + i$. Define a branch of the logarithm which is analytic on a domain containing $C$ and use it to evaluate $\int_C \frac{dz}{z}$. This problem requires both a number and a function.

There was a typo in the original, and you can actually use the Principal Value of the logarithm here. (What I meant will be on homework 7.). Since the line segments do not cross the negative real axis, we just wind up with $\text{Log}(2 + i) - \text{Log}(1 - 2i)$; that is,

$$\ln(\sqrt{5}) + i \arctan(1/2) - \ln(\sqrt{5}) - i \arctan(-2) = \frac{i \pi}{2}.$$