1. §1.1 – 6. In general, \((r, \theta)\) is associated with \(z = r \cos \theta + i r \sin \theta\), so the answers are;
(a) \(\sqrt{3} (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) = \sqrt{3} (\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}) = \frac{\sqrt{3}}{2} (1 + i)\).
(b) \(\frac{1}{\sqrt{2}} (\cos(- \pi) + i \sin(- \pi)) = -\frac{\sqrt{2}}{2}\).
(c) \(4 (\cos(- \frac{\pi}{4}) + i \sin(- \frac{\pi}{4})) = 4 (0 - i) = -4i\).
(d) \(2 (\cos(- \frac{\pi}{4}) + i \sin(- \frac{\pi}{4})) = 2 (\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}) = \sqrt{2} (1 - i)\).
(e) \(\cos(4\pi) + i \sin(4\pi) = 1\).
(f) \(\sqrt{2} (\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4}) = \sqrt{2} (\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}) = 1 + i\).

2. §1.1 – 18. Inductions require both an initial step and an inductive step. The induction might properly begin at \(n = 0\) or \(n = 1\) or \(2\), since the “…” is a suggestion of more than one term on the left. The author meant the formula to be valid for \(n \geq 0\), so I’ll start there.

If \(n = 0\), then the given formula says that \(1 = \frac{1 - z}{1 - z}\), and this is true for \(z \neq 1\). If the formula is valid for \(n\), then we can write down the lefthand side for \(n + 1\) and then replace most of it with the inductive formula, yielding the desired inductive step:

\[
1 + z + \cdots + z^n + z^{n+1} = \frac{1 - z^{n+1}}{1 - z} + z^{n+1} = \frac{1 - z^{n+1} + z^{n+1} (1 - z)}{1 - z} = \frac{1 - z^{n+2}}{1 - z}.
\]

3. §1.2 – 2. Just square to eliminate the square roots. If \(z = x + iy\), then

\[
|z - 4| = 4|z| \iff |z - 4|^2 = 16|z|^2 \iff (x - 4)^2 + y^2 = 16(x^2 + y^2) \iff 0 = 15x^2 + 8x - 16 + 15y^2 \iff 15(x + \frac{4}{15})^2 + 15y^2 = 16 + 15(\frac{4}{15})^2 = \frac{256}{15}.
\]

This means that the locus is a circle with center \((-\frac{4}{15}, 0)\) and radius \(\sqrt{\frac{256}{15}} = \frac{16}{15}\).

4. §1.2 – 16. I don’t know what I was thinking with the hint, and this explains the question in class on Wednesday. The intended answer here is simply \(|z - (-1 + 2i)| = |z - (1 - 2i)|\).

5. §1.2 – 24 and 26. If \((z + 1)^4 = 1 - i = \sqrt{2} e^{-\frac{\pi}{4}}\), then \(z + 1\) has modulus \(2^{1/8}\), and one of four arguments: \(\frac{1}{4}(-\frac{\pi}{4}) + k\frac{2\pi}{4}\). The answers are thus

\[
2^{1/8} e^{-\frac{\pi}{4}} - 1, \quad 2^{1/8} e^{\frac{\pi}{4}} - 1, \quad 2^{1/8} e^{\frac{3\pi}{4}} - 1, \quad 2^{1/8} e^{\frac{5\pi}{4}} - 1.
\]

For 26., \(8 = 8e^0 i\), and \(8^{1/3} = 2\), hence the three cube roots are

\[
\{2, 2e^{\frac{2\pi}{3}}, 2e^{\frac{4\pi}{3}}\} = \{2, -1 + i\sqrt{3}, -1 - i\sqrt{3}\}
\]

6. §1.3 – 3. I meant to assign 4., but didn’t catch the typo. The set \(C\) in the plane is the open region above the parabola \(y = x^2\). I would say that \(C\) is its own interior and its boundary is the parabola itself. The set is open and the interior is connected; in fact, it’s convex.
7. (E) Find explicit real numbers $r$ and $s$ with the property that

$$\frac{1}{2^{173}} \left(\frac{5 + 3i}{1 + 4i}\right)^{348} = r + is.$$

First we rationalize the inner fraction:

$$\frac{5 + 3i}{1 + 4i} = \frac{(5 + 3i)(1 - 4i)}{(1 + 4i)(1 - 4i)} = \frac{17 - 17i}{17} = 1 - i = \sqrt{2} \cdot e^{-\frac{i\pi}{4}}.$$

Thus, we have

$$r + is = \frac{1}{2^{173}} \left(\sqrt{2} \cdot e^{-\frac{i\pi}{4}}\right)^{348} = \frac{2^{174}}{2^{173}} \cdot e^{-87i\pi} = -2.$$

8. (E) Sketch the region $R$, consisting of the complex numbers $re^{i\theta}$ with $1 \leq r \leq 2$ and $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$, and also sketch its range under the mapping $w = \frac{1}{z}$. See sketch. Note that if $w = re^{i\alpha}$, then $\rho = 1/r$ and $\alpha = -\theta$ (up to a multiple of $2\pi$). Thus $\frac{1}{2} \leq \rho \leq 1$ and $-\frac{\pi}{4} \geq \rho \geq -\frac{3\pi}{4}$.

9. §1.2 – 30. Let $z = x + iy$ and $A = a + ib$, where $x, y, a, b$ are real. Then the given equation expands as follows:

$$0 = |z|^2 + Re(Az) + B = x^2 + y^2 + (ax - by) + B.$$  

This sort of equation is dealt with by completing the square. We have

$$(x + \frac{a}{2})^2 + (y - \frac{b}{2})^2 = \frac{a^2 + b^2 - 4B}{4} = \frac{|A|^2 - 4B}{4},$$

which has (real) solutions in $(x, y)$ if and only if $|A|^2 \geq 4B$. These solutions are either a point (if $|A|^2 = 4B$) or a circle (if $|A|^2 > 4B$).

10a. (E) Find all pair of complex numbers $(\alpha, \beta)$ with the property that if $w = \alpha z + \beta$, then the line $Im(z) = 1$ is mapped to the line $Re(w) = 2$.

Note: I intended “to” to be “onto” (this is a conventional elision), but was a bit sloppy in the writing. If $\alpha = 0$ and $\beta$ is any complex number whose real part is 2, then the condition is satisfied, and I guess I have to accept it for (a). In general, what one does
is to write out the variable and the conditions explicitly. If $\text{Im}(z) = 1$, then $z = x + i$ for some real $x$. If we then write $\alpha = a + bi$ and $\beta = c + di$ with real $a, b, c, d$, then

$$\alpha z + \beta = (a + ib)(x + i) + (c + id) = (ax - b + c) + i(a + bx + d).$$

The desired condition is that the real part be 2; that is, that $ax - b + c = 2$ for all $x$. This implies that $a = 0$ and $c = b + 2$. In other words, the solution is

$$\alpha = bi, \quad \beta = (b + 2) + di,$$

where $b$ and $d$ are arbitrary reals.

This can be interpreted geometrically as follows: $w = biz + (2 + b + di) = b(z + 1) + 2 + di$. So first, $z \mapsto iz$, which takes the line $\text{Im}(z) = 1$ to $\text{Re}(z) = -1$ (check this yourself!). Then $iz \mapsto iz + 1$, which takes $z$ to the imaginary axis $\text{Re}(z) = 0$. Then $iz + 1 \mapsto b(iz + 1)$, which is an arbitrary real scaling that maps the imaginary axis to itself. Finally, $b(iz + 1) \mapsto b(iz + 1) + (2 + di)$, which is a translation to the right by 2 units (to $|\text{Re}(z) = 2$ and up(or down) by an arbitrary amount ($d$ units).

Bonus question: Define integers $a_n$ and $b_n$ by $a_n + ib_n = (3 + 4i)^n$. Prove that, for all $n \geq 1$, $a_n \equiv 3$ (mod 5) and $b_n \equiv 4$ (mod 5). That is, there exist integers $c_n$ and $d_n$ so that $a_n = 5c_n + 3$ and $b_n = 5d_n + 4$. (Hint: $a_{n+1} + ib_{n+1} = (3 + 4i)(a_n + ib_n)$.) Explain why this implies that $(3 + 4i)^n$ is never real, and hence why $\frac{\arctan 4/3}{\pi}$ is irrational.

Writing $a_n + ib_n = (3 + 4i)^n$, we have

$$a_{n+1} + ib_{n+1} = (a_n + ib_n)(3 + 4i) = (3a_n - 4b_n) + i(4a_n + 3b_n).$$

If you know modular arithmetic, it’s easy to see that $a_n \equiv 3$ mod 5 and $b_n \equiv 4$ mod 5 imply the same for $a_{n+1}$ and $b_{n+1}$. If you don’t know modular arithmetic, prove it by induction, by writing $a_n = 5c_n + 3$ and $b_n = 5d_n + 4$, and assume that $c_n$ and $d_n$ are integers. Then $c_0 = d_0 = 0$, and the recurrence gives

$$5c_{n+1} + 3 = 3(5c_n + 3) - 4(5d_n + 4) = 5(3c_n - 4d_n - 2) + 3 \implies c_{n+1} = 3c_n - 4d_n - 2$$
$$5d_{n+1} + 4 = 4(5c_n + 3) + 3(5d_n + 4) = 5(4c_n + 3d_n + 4) + 4 \implies d_{n+1} = 4c_n + 3d_n + 4.$$ 

Thus, if $c_n$ and $d_n$ are integers, so are $c_{n+1}$ and $d_{n+1}$.

If $\frac{\arctan 4/3}{\pi} = \frac{m}{n}$, then $z = 3 + 4i = 5e^{m\pi i/n}$, so that $z^n = (3 + 4i)^n = 5^n e^{m\pi i} = \pm 5^n$ would be real, and so that $b_n = 0$, but this is impossible since $b_n \equiv 4 \mod 5$.

A more general theorem states that $\tan \frac{m\pi}{n}$ is a rational number only when $\frac{m\pi}{n}$ is a multiple of $\frac{\pi}{4}$. This is a relatively simple consequence of Galois Theory, and was first proved by Ivan Niven. It follows that if $a$ and $b$ are non-zero integers and $|a| \neq |b|$, then $(a + ib)^n$ is never real for integral $n \geq 1$. 

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