1. - 17.1 (ungraded); 3. 17.9 (ungraded). Answers in back. Ask if you have a question.

2. - 17.2. Here, \( f(x) = 4 \) if \( x \geq 0 \) and \( f(x) = 0 \) if \( x < 0 \) and \( g(x) = x^2 \) for all real \( x \), and \( \text{dom}(f) = \text{dom}(g) = \mathbb{R} \).

(a) Then \( (f + g)(x) = 4 + x^2 \) if \( x \geq 0 \) and \( (f + g)(x) = x^2 \) if \( x < 0 \); \( (fg)(x) = 4x^2 \) if \( x \geq 0 \) and \( (fg)(x) = 0 \) if \( x < 0 \); \( (f \circ g)(x) = f(g(x)) = f(x^2) = 4 \) (for all \( x \)) and \( (g \circ f)(x) = g(f(x)) = (f(x))^2 = 16 \) if \( x \geq 0 \) and \( (g \circ f)(x) = 0 \) if \( x < 0 \). In each of these cases, the domain is \( \mathbb{R} \).

(b) Recall that polynomials are continuous and that if \( f \) jumps at \( x = a \), then it is not continuous at \( a \), because, if \( s_n \to a \) and \( s_n < a \) and \( t_n \to a \), but \( t_n > a \), then \( (f(s_n)) \) and \( (f(t_n)) \) will converge to different values. On this basis, \( g \), and \( f \circ g \) are polynomials, and so are continuous, but \( f \), \( f + g \) and \( g \circ f \) jump at \( 0 \), and so are not continuous there. Note that \( fg \) is not a polynomial, but it doesn’t have a jump at \( x = 0 \), so it’s continuous. (See Homework 8, #8.)

4. - 18.6. Assuming that \( \cos x \) is continuous, let \( f(x) = \cos x - x \), which is also continuous. We have \( f(0) = \cos 0 - 0 = 1 \) and \( f(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) - \frac{\pi}{2} = -\frac{\pi}{2} \). Thus, \( f(0) > 0 > f(\frac{\pi}{2}) \). By the Intermediate Value Theorem, it follows that there exists \( x_0 \in [0, \frac{\pi}{2}] \) so that \( f(x_0) = \cos x_0 - x_0 = 0 \).

5. - 18.10. So, suppose \( f(0) = f(2) \) and \( f \) is continuous on \([0, 2]\). Let \( g(x) = f(x+1) - f(x) \). Then \( g \) is continuous on \([0, 1]\) and \( g(0) + g(1) = f(2) - f(0) = 0 \). Thus, either \( g(0) = g(1) = 0 \), which means that \( f(0) = f(1) = f(2) \), or \( \{g(0), g(1)\} \) consists of one positive and one negative number. No matter which, the IVT implies that there exists \( x \in [0, 1] \) so that \( g(x) = 0 \); that is, \( f(x) = f(x+1) \).

6. - 19.2a. Observe that \( |f(x) - f(y)| = |(3x + 11) - (3y + 11)| = 3|x - y| \). So, given \( \epsilon > 0 \), we see that

\[
|x - y| < \frac{\epsilon}{3} \implies |f(x) - f(y)| = 3|x - y| < \epsilon.
\]

7. - 19.2b. With \( f(x) = x^2 \) on \([0, 3]\), we have

\[
|f(x) - f(y)| = |x^2 - y^2| = |x - y| \cdot |x + y| \leq |x - y| \cdot (3 + 3) = 6|x - y|.
\]

So, given \( \epsilon > 0 \), we see that

\[
|x - y| < \frac{\epsilon}{6} \implies |f(x) - f(y)| \leq 6|x - y| < 6 \cdot \frac{\epsilon}{6} = \epsilon.
\]

8. - 19.6. My hint is for a really fast way to do a. and b. at once. I’ll start with an acceptable solution to the problem that ignores the hint.
a. If \( f(x) = \sqrt{x} \), then \( f'(x) = \frac{1}{2\sqrt{x}} \) is unbounded on \((0, 1]\). This is obvious, but it’s also easily proved: if \( x_n = n^{-2} \in (0, 1] \), then \( f'(x_n) = n/2 \), which is unbounded as \( n \to \infty \). On the other hand, \( f \) is the inverse of the (continuous, monotone) polynomial \( x^2 \), so it is continuous by Theorem 18.4. Theorem 19.2 says that if \( f \) is continuous on a closed interval \([a, b]\), then \( f \) is uniformly continuous on \([a, b]\), so \( f \) is uniformly continuous on \([0, 1]\).

b. To show that \( f \) is uniformly continuous on \([1, \infty)\), observe that

\[
\sqrt{x} - \sqrt{y} = \frac{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}{\sqrt{x} + \sqrt{y}} = \frac{x - y}{\sqrt{x} + \sqrt{y}}.
\]

If \( x, y \geq 1 \), then \( \sqrt{x}, \sqrt{y} \geq 1 \), so the denominator above is at least 2 and

\[
|\sqrt{x} - \sqrt{y}| \leq \frac{|x - y|}{2}.
\]

Thus, if \( \epsilon \) is given, then \( |x - y| < 2\epsilon \) \( \implies \) \( |f(x) - f(y)| < \epsilon \), and so \( f \) is uniformly continuous.

Here is my proof that \( f \) is uniformly continuous on \([0, \infty)\), and this subsumes the results above.

**Lemma:** Fix \( t > 0 \). Let \( \Phi_t(x) = \sqrt{x + t} - \sqrt{x} \) for \( x \geq 0 \). Then \( \Phi_t \) is strictly decreasing.

**Proof:** Observe that

\[
\Phi_t(x) = \sqrt{x + t} - \sqrt{x} = \frac{1}{\sqrt{x + t} + \sqrt{x}}.
\]

Since \( x < y \) clearly implies \( \sqrt{x + t} + \sqrt{x} < \sqrt{y + t} + \sqrt{y} \), it follows that \( \Phi_t(x) > \Phi_t(y) \). Taking \( y = 0 \), we see that \( \sqrt{x + t} - \sqrt{x} < \sqrt{t} - \sqrt{0} = \sqrt{t} \). Putting \( t = x - y \), we find the key inequality

\[
\sqrt{y} - \sqrt{x} < \sqrt{y - x}.
\]

(This can also be shown directly, by squaring both sides and transposing.)

Now, suppose \( x, y \geq 0 \) and \( |x - y| < \delta \). Without loss of generality, suppose that \( x \geq y \), so \( \sqrt{x} \geq \sqrt{y} \). Then \( \sqrt{x} - \sqrt{y} < \sqrt{x - y} < \sqrt{\delta} \). Thus, if \( \epsilon > 0 \) is given, then

\[
|x - y| < \epsilon^2 \implies |\sqrt{x} - \sqrt{y}| < \epsilon.
\]

The proof applies to any function \( f \) which is a monotone increasing continuous function on \([0, \infty)\) satisfying \( f(0) = 0 \) and so that \( x \geq y \) implies \( f(x) - f(y) \leq f(x - y) \).

---

9. – 17.14. This is actually a famous function \( f \) so that, if \( x \in \mathbb{Q} \) is given in lowest terms as \( x = \frac{p}{q} \), then \( f(x) = \frac{1}{q} \), and if \( x \) is irrational, then \( f(x) = 0 \). We have to show that \( f \) is continuous at each irrational \( x \) and not continuous at each rational \( x \).

For the first, suppose \( x \) is irrational and suppose \( \epsilon < 1 \) is given. Choose \( N < \frac{1}{\epsilon} \). The interval \((x - 1, x + 1)\) contains only finitely many rational numbers with denominator \( \leq N \). (In fact, each denominator \( r \) occurs less than \( 2r \) times.) None of these rational numbers
equals \( x \), so there exists \( \delta > 0 \) so that if \( |y - x| < \delta \) and \( y \) is rational, then the denominator of \( y \) is \( \geq N \). This means that \( f(y) \leq \frac{1}{N} < \epsilon \) if \( y \) is rational, and of course \( f(y) = 0 \) if \( y \) is irrational. Therefore, \( |x - y| < \delta \) implies that \( |f(x) - f(y)| < \epsilon \), and this means that \( f \) is continuous at \( x \).

If \( x \) is rational, then \( x_n = x + \frac{1}{n} \sqrt{2} \) is irrational and \( \lim x_n = x \). But \( f(x_n) = 0 \) for all \( n \), so \( \lim f(x_n) = 0 \neq f(x) \), and so \( f \) is not continuous at \( x \).

One purpose of this mind-bogglingly non-intuitive example is to show how mind-bogglingly non-intuitive real analysis can be.

10a. – This is known as the “Cauchy Condensation Theorem”. Suppose \( (a_n) \) is a decreasing sequence of positive real numbers, and let \( b_n = a_{2^n} \). Prove that \( \sum a_n \) is convergent if and only if \( \sum 2^n b_n \) is convergent. (Hint: the proof of the \( p \)-test on Bonus Notes 8.)

Let \( s_m = \sum_{n=1}^{m} a_n \). Then \( (s_m) \) is an increasing sequence, so it’s either bounded above or diverges to \( \infty \). Following the hint, observe that, if \( 2^r \leq n < 2^{r+1} \), then \( a_{2^r} \geq a_n \geq a_{2^{r+1}} \); that is, \( b_r \geq a_n \geq b_{r+1} \). Thus, we have for this block of \( 2^{r+1} - 2^r = 2^r \) terms

\[
2^r b_r \geq a_{2^r} + \cdots + a_{2^{r+1}-1} > 2^r b_{r+1},
\]

and so, summing from \( r = 0 \) to \( N - 1 \),

\[
\sum_{r=0}^{N-1} 2^r b_r \geq \sum_{r=0}^{N-1} (a_{2^r} + \cdots + a_{2^{r+1}-1}) = \sum_{n=1}^{N-1} a_n \geq \sum_{r=0}^{N-1} 2^r b_{r+1} = \frac{1}{2} \sum_{r=1}^{N} 2^r b_r.
\]

If \( \sum 2^n b_n \) is convergent, then the left hand side is bounded above, and so \( \sum a_n \) is convergent. If \( \sum 2^n b_n \) is divergent, then the right hand side is unbounded and so \( \sum a_n \) is divergent. With \( a_n = n^{-p} \), \( 2^n b_n = 2^n (2^n)^{-p} = (2^{1-p})^n \), and the ratio or root test shows convergence if \( p > 1 \) and divergence if \( p \leq 1 \).

10b. Use (a) to determine the values of \( p \) for which

\[
\sum \frac{1}{n (\log n)^p}
\]

converges. We want to use (a), so we need to check that this is decreasing, or, equivalently, that \( \phi(x) = x (\log x)^p \) is increasing. Notice that if \( p < 0 \), then the series diverges with comparison to \( \sum \frac{1}{n} \), so we can assume that \( p \geq 0 \), and then \( \phi'(x) = (\log x)^p + px (\log x)^{p-1} (1/x) \geq 0 \), at least for \( x > e \), which is good enough. Using the preceding, we have

\[
2^n a_{2^n} = \frac{2^n}{2^n (\log 2^n)^p} = \frac{1}{(n \log 2)^p} = \frac{1}{(\log 2)^p} \cdot \frac{1}{n^p}.
\]

By the \( p \)-test, this is convergent if \( p > 1 \) and divergent if \( p \leq 1 \). This problem can also be done by the integral test; noting that

\[
\int \frac{dx}{x (\log x)^p} = \begin{cases} \frac{1}{1-p} \cdot \frac{1}{(\log x)^{p-1}} + C, & \text{if } p \neq 1, \\ \log \log x + C, & \text{if } p = 1. \end{cases}
\]