1. and 4. – 14.3 – a,b,c,d and 14.7 (ungraded) See answers in back. Ask if unclear.

2a. Observe that the dominant term is \( n \) and \( n + (-1)^n \geq n - 1 \), hence for \( n \geq 2 \),
\[
\frac{1}{(n + 1)^n} \leq \frac{1}{(n - 1)^n}.
\]
Since \( \sum \frac{1}{(n - 1)^n} \) is convergent (by an index-shift and the \( p \)-test with \( p = 2 \), or by \( (n - 1)^2 \geq n^2/4 \) or by the theorem in class and on Bonus Notes 9, considering \( \frac{p(n)}{q(n)} \) with \( p(x) = 1 \) and \( q(x) = (x - 1)^2 \), this series is convergent.

b. There are two ways to do this. One is to let \( a_n = \sqrt{n + 1} - \sqrt{n} \) and then observe that the partial sums telescope to \( s_n = \sqrt{n + 1} - 1 \). Since \( n \to \infty \), the series diverges.

The other way to do this is to exploit the algebraic identity:
\[
\sqrt{n + 1} - \sqrt{n} = \left( \frac{\sqrt{n + 1} + \sqrt{n}}{\sqrt{n + 1} + \sqrt{n}} \right) = \frac{1}{\sqrt{n + 1} + \sqrt{n}}.
\]
This suggests divergence, and in fact, if you use the inequality \( \sqrt{n + 1} - \sqrt{n} > \frac{1}{2} \), then the comparison test and the \( p \)-test with \( p = \frac{1}{2} \) shows divergence.

c. Using the hint of the root test, we have \( \left( \frac{n!}{n^n} \right)^{1/n} = \frac{(n!)^{1/n}}{n} \to \frac{1}{e} < 1 \) as shown in class, so the series converges. If you forgot that, you can still use the ratio test, and obtain
\[
\frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n+1)!}{(n+1) \cdot n!} \cdot \frac{n^n}{(n+1)^n} = \left( \frac{n}{n+1} \right)^n \to \frac{1}{e}.
\]

(This computation was how we got the first limit anyway.)

3. – 14.6a. Suppose \( \sum |a_n| \) converges and \( (b_n) \) is bounded, say \( |b_n| \leq M \) for all \( n \). (a) Show that \( \sum a_n b_n \) converges. Following the hint, we’ll prove that it’s Cauchy. Indeed, since \( \sum |a_n| \) converges, it is Cauchy, and for every \( \epsilon > 0 \) there exists \( N \) so that for \( m, n > N \), \( \sum_{k=m+1}^{n} |a_k| < \epsilon \). So, given \( \epsilon > 0 \), there exists \( N' \) so that \( m, n > N' \), \( \sum_{k=m+1}^{n} |a_k| < \epsilon/M \). We do this so we can make the following estimate, showing that \( \sum a_n b_n \) is Cauchy:
\[
\left| \sum_{k=m+1}^{n} a_k b_k \right| \leq \sum_{k=m+1}^{n} |a_k| |b_k| \leq \left( \sum_{k=m+1}^{n} M \cdot |a_k| \right) < M \cdot \frac{\epsilon}{M} = \epsilon.
\]

The point in (b) is that \( a_n = (\pm 1)|a_n| \), depending on the sign of \( a_n \), and a sequence where \( b_n = \pm 1 \) is bounded.

5. – 14.8. Use the hint! Suppose \( x \) and \( y \) are non-negative numbers. We have
\[
0 \leq (\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} + y \implies \sqrt{xy} \leq \frac{x+y}{2} \leq x+y.
\]
Let \( (s_n) \) and \( (t_n) \) denote the partial sums of \( (a_n) \) and \( (b_n) \) respectively. In view of the hint, and because \( a_n, b_n \geq 0 \), we have
\[
\sum_{k=1}^{n} \sqrt{a_k b_k} \leq \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = s_n + t_n.
\]
Since \((s_n)\) and \((t_n)\) are bounded non-decreasing sequences, so is the sequence of partial sums for the given series, and hence it is convergent.

6. – 14.12. Given a sequence \((a_n)\) so that \(\lim \inf |a_n| = 0\), we wish to find a subsequence \(a_{n_k}\) so that \(\sum_{k=1}^{\infty} a_{n_k}\) is convergent. In a problem like this, since the information is about \(|a_n|\), you want to prove that the sum of the subsequence is absolutely convergent, and the easiest thing is to find a subsequence so that \(a_{n_k} \leq \frac{1}{2^k}\). How can this be done? We know that \(\lim \inf |a_n| = 0\); thus if \(u_N = \inf\{|a_n| : n > N\}\), then \(u_N \geq 0\) for all \(n\) (since 0 is a lower bound) and \((u_n)\) is non-decreasing, hence \(u_N = 0\) for all \(N\). Thus, for all \(k\) and all \(N\), \(2^{-k} > u_N\), thus there exists \(n > N\) so that \(2^{-k} > |a_n|\). So here’s what we do. We pick \(n_1\) so that \(|a_{n_1}| \leq 1/2\). Then we pick \(n_2 > n_1\) so that \(|a_{n_2}| \leq 1/4\), etc. This gives the desired subsequence so that \(|a_{n_k}| \leq 2^{-k}\), and \(\sum a_{n_k}\) is absolutely convergent and hence is convergent.

7. – 16.4 - c.f. We have, for the first,

\[
.0\overline{2} = \frac{0}{10^1} + \frac{2}{10^2} + \frac{0}{10^3} + \frac{2}{10^4} + \cdots = 2 \sum_{k=1}^{\infty} \frac{1}{10^{2k}} = \frac{2}{1 - \frac{1}{100}} = \frac{2}{99}.
\]

For the second, a little more care is required,

\[
.\overline{492} = \frac{1}{10^1} + \frac{4}{10^2} + \frac{9}{10^3} + \frac{2}{10^4} + \frac{4}{10^5} + \frac{9}{10^6} + \frac{2}{10^7} + \cdots = \frac{1}{10^1} + \frac{492}{10^4} + \frac{492}{10^7} + \cdots
\]

\[
= \frac{1}{10^1} + \frac{492}{10^4} \cdot \left( \sum_{k=0}^{\infty} \frac{1}{10^{3k}} \right) = \frac{1}{10^1} + \frac{492}{10^4} \cdot \frac{1000}{999} = \frac{1}{10} + \frac{492}{9990} = \frac{1491}{9990}.
\]

8. – (The correction.) Suppose \((s_n)\) and \((t_n)\) are bounded sequences, but not necessarily non-negative and not necessarily convergent, and suppose \(\limsup s_n = s\) and \(\limsup t_n = t\). Is it a correct theorem that \(\limsup (s_n + t_n) \leq s + t\)? Is it a correct theorem that \(\limsup (s_n t_n) \leq st\)? This problem requires either a proof similar to that on the last homework, or a counterexample or both.

I think I've made much too big a deal out of this. The theorem is correct for the sum and false for the product. Following the old proof, given \(\epsilon > 0\), \(\limsup s_n = s\) implies that there exists \(N_1\) so that \(\sup\{s_n : n > N_1\} < s + \epsilon/2\), hence for each \(n > N_1\), \(s_n < s + \epsilon/2\). Similarly, there is \(N_2\) so that for \(n > N_2\), \(t_n < t + \epsilon/2\), and thus, if \(N = \max\{N_1, N_2\}\), then for \(n > N\), \(s_n + t_n < s + t + \epsilon\). It follows that \(\limsup (s_n + t_n) < s + t + \epsilon\). This is true for every \(\epsilon > 0\) and so \(\limsup (s_n + t_n) < s + t\). The point here is that non-negativity isn't a factor when you're adding.

For multiplication, it's a different story. The simplest counterexample has \((s_n) = (t_n)\) being the sequence alternating between 0 and -1. Then \(s = t = 0\), but \((s_n t_n)\) is the sequence alternating between 0 and \((-1)^2 = 1\), so \(\limsup (s_n t_n) = 1\), which is not \(\leq 0\).

9. – Observe that for every \(n \in N\), there exists \(r \geq 0\) so that \(2^r \leq n \leq 2^{r+1} - 1\). (In fact, \(r = \lfloor \log_2 n \rfloor\).) Define a sequence \((s_n)\) as follows: if \(2^r \leq n \leq 2^{r+1} - 1\) and \(r\) is even, then
$s_n = 1$; if $r$ is odd, then $s_n = 0$. Thus, for $n = 1, 2, 3, 4, 5, 6, 7$, we have $r = 0, 1, 1, 2, 2, 2$ and $s_n = 1, 0, 0, 1, 1, 1, 1$. As before, let $\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n)$. Compute $\sigma_n$ when $n = 2^r + j$, $0 \leq j \leq 2^r - 1$ and determine $\lim\inf \sigma_n$ and $\lim\sup \sigma_n$. (Hint: first compute $\sigma_{2^k}$ and $\sigma_{2^{2k+1}}$.)

Perhaps a better hint would have been to compute $\sigma_{2^{r-1}}$. Let $x_n = s_1 + \cdots + s_n$.

Then $x_1 = 1$, $x_3 = 1$, $x_7 = 5$ from the information given above, and $x_{15} = 5$ as well, because for $8 \leq n \leq 15$, we have $r = 3$ and $s_n = 0$. But for $16 \leq n \leq 31$, we have $r = 4$ and $s_n = 1$, so that $x_{31} = 5 + 16 = 21$, and for $32 \leq n \leq 63$, we have $r = 5$ and $s_n = 0$, so $x_{63} = 21$ as well. I hope you can see the pattern: $x_n = x_{n-1} + 1$ if and only if $r$ is even, so there are blocks in which $x_n$ is incrementing by 1 and blocks in which it is constant.

$x_1 = x_3 = 1; x_4 = 2, \ldots, x_6 = 4x_7 = x_{15} = 5; x_{16} = 6, \ldots, x_{30} = 20, x_{31} = x_{63} = 21; \ldots$

The relevance of 1, 5, 21 is that they are 1, 1 + 4, 1 + 4 + 16. In fact, it’s easy to see that

$$x_{2^{2k+1} - 1} = x_{2^{2k+2} - 1} = 1 + 4 + \cdots + 4^k = \frac{4^{k+1} - 1}{4 - 1} \approx \frac{2}{3} \cdot 2^{2k+1}.$$ 

It follows from this that $\sigma_{2^{2k+1} - 1} \to \frac{2}{3}$ and $\sigma_{2^{2k+2} - 1} \to \frac{1}{3}$.

What about the other $\sigma_n$’s? Well, notice that $s_n = 0$ or 1 and so $0 \leq \sigma_n \leq 1$. Furthermore, if $s_n = 1$, then $\sigma_n \leq \sigma_{n+1}$ and if $s_n = 0$, then $\sigma_n \geq \sigma_{n+1}$. We see then that $\sigma_n$ is increasing from $n = 2^{2k} - 1$ to $n = 2^{2k+1} - 1$ and decreasing from $n = 2^{2k+1}$ to $n = 2^{2k+2} - 1$, so the values found above in fact show that $\lim\sup \sigma_n = \frac{2}{3}$ and $\lim\inf \sigma_n = \frac{1}{3}$.

10. - Construct a sequence $(s_n)$ so that

$$\lim\inf \frac{s_{n+1}}{s_n} < \lim\inf \frac{s_n^{1/n}}{s_n} < \lim\sup \frac{s_n^{1/n}}{s_n} < \lim\sup \frac{s_{n+1}}{s_n}$$

Suggestion: Emulate the example of Homework 5, #7, but make different rules for $\frac{s_{n+1}}{s_n}$ depending on whether $2^k \leq n < 2^{k+1}$ or $2^{k+1} \leq n \leq 2^{k+2}$. Suppose we follow the pattern of problem 9, and say that $\frac{s_{n+1}}{s_n} = 2$ if $2^k \leq n < 2^{k+1}$ and $\frac{s_{n+1}}{s_n} = 1$ if $2^{k+1} \leq n < 2^{k+2}$. Then $\frac{s_{n+1}}{s_n}$ takes the values 1 or 2 infinitely often, hence their liminf and limsup are 1 and 2 respectively.

What is $s_n$? Nothing less that $2^{x_n}$ from the last problem! What is $s_n^{1/n}$? Nothing less that $2^{x_n}$ from the last problem! Therefore, we have

$$\lim\inf \frac{s_{n+1}}{s_n} = 1 < \lim\inf \frac{s_n^{1/n}}{s_n} = 2^{1/3} < \lim\sup \frac{s_n^{1/n}}{s_n} = 2^{2/3} < \lim\sup \frac{s_{n+1}}{s_n} = 2.$$ 

Naturally, other examples are possible and will be carefully considered.