GEOMETRICAL ILLUSTRATIONS.

straight line and a parallel to it through the origin will be of constant length; (4), that a straight line is nowhere either convex or concave—and this property, which does not involve, in the same definite manner as the others do, the considerations of distance and of angular magnitude, is evidently the most absolute of the three.

The equation of the circle is

\[(a-a)^2 + (y-b)^2 = r^2 \ldots \ldots \ldots \ldots \ldots (5),\]

and if we regard \(a\) and \(b\) as arbitrary constants the corresponding differential equation of the second order will be

\[\left\{1 + \left(\frac{dy}{dx}\right)\right\}^2 \frac{d^2 y}{dx^2} = r \ldots \ldots \ldots \ldots \ldots (6),\]

expressing the property that the radius of curvature is invariable and equal to \(r\).

If we proceed to another differentiation, we find

\[\left\{1 + \left(\frac{dy}{dx}\right)^2 \right\} \frac{d^3 y}{dx^3} - 3 \frac{dy}{dx} \left(\frac{d^2 y}{dx^2}\right)^2 = 0 \ldots \ldots (7),\]

which is the general differential equation of a circle free from arbitrary constants. And the geometrical property which this equation also expresses is the invariability of the radius of curvature, but the expression is of a more absolute character than that of the previous equation (6). For in that equation we may attribute to \(r\) a definite value, and then it ceases to be the differential equation of all circles, and pertains to that particular circle only whose radius is \(r\). The equation (7) admits of no such limitation.

Monge has deduced the general differential equation of lines of the second order expressed by the algebraic equation

\[\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \epsilon y = 1.\]
It is
\[ 9 \left( \frac{dy}{dx} \right)^2 \frac{d^3 y}{dx^3} - 45 \frac{d^2 y}{dx^2} \frac{d^4 y}{dx^4} + 40 \left( \frac{d^3 y}{dx^3} \right)^2 = 0. \]

But here our powers of geometrical interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms.

From the above examples it will be evident that the higher the order of the differential equation obtained by elimination of the determining constants from the equation of a curve, the higher and more absolute is the property which that differential equation expresses.

We reserve to a future Chapter the consideration of the genesis of partial differential equations as well as of ordinary differential equations involving more than two variables.

**EXERCISES.**

1. Distinguish the following differential equations according to species, order, and degree, and take account of any peculiarities dependent upon their coefficients.

   (1) \[ \frac{dy}{dx} - \alpha y = \alpha x. \]

   (2) \[ \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} - \alpha y = 0. \]

   (3) \[ y = \omega + \omega \frac{dy}{dx} + \left( \frac{dy}{dx} \right)^2. \]

   (4) \[ \alpha \frac{ds}{dx} - y \frac{ds}{dy} = \frac{s}{y}. \]

   (5) \[ \frac{du}{dx} + \frac{d^2 u}{dy^2} + \frac{d^2 u}{dx^2} = 0. \]