Blenders

Bruce Reznick

Dedicated to the memory of Julius Borcea

Abstract. A blender is a closed convex cone of real homogeneous polynomials that is also closed under linear changes of variable. Non-trivial blenders only occur in even degree. Examples include the cones of psd forms, sos forms, convex forms and sums of $2u$-th powers of forms of degree v. We present some general properties of blenders and analyze the extremal elements of some specific blenders.

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1. Introduction and overview

Let $F_{n,d}$ denote the vector space of real homogeneous forms $p(x_1, \dots, x_n)$ of degree d. A blender is a closed convex cone in $F_{n,d}$ that is also closed under linear changes of variable. Blenders were introduced in [19] to help describe several different familiar cones of polynomials, but that memoir was mainly concerned with the cones of psd and sos forms and their duals, and the discussion of blenders *per se* was scattered (pp. 36-50, 119-120, 140-142). This paper is devoted to a general discussion of blenders and their properties, as well as the extremal elements of some particular blenders not discussed in [19].

We shall see that non-trivial blenders only occur when $d = 2r$ is an even integer. Choi and Lam [4, 5] named the cone of *psd* forms:

$$P_{n,2r} := \{p \in F_{n,2r} : u \in \mathbb{R}^n \implies p(u) \geq 0\}, \tag{1.1}$$

and the cone of *sos* forms:

$$\Sigma_{n,2r} := \left\{ p \in F_{n,2r} : p = \sum_{k=1}^{s} h_k^2,\ h_k \in F_{n,r} \right\}. \tag{1.2}$$

Other blenders of interest in [19] are the cone of sums of $2r$-th powers:

$$Q_{n,2r} := \left\{ p \in F_{n,2r} : p = \sum_{k=1}^{s} (\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2r},\ \alpha_{kj} \in \mathbb{R} \right\} \tag{1.3}$$

and the "Waring blenders". Suppose $r = uv$, $u, v \in \mathbb{N}$ and let:

$$W_{n,(u,2v)} := \left\{ p \in F_{n,2r} : p = \sum_{k=1}^{s} h_k^{2v},\ h_k \in F_{n,u} \right\}. \tag{1.4}$$

Note that $W_{n,(r,2)} = \Sigma_{n,2r}$ and $W_{n,(1,2r)} = Q_{n,2r}$.

The Waring blenders generalize. If $d = 2r$ and $\sum_{i=1}^{m} u_i v_i = r$, let

$$W_{n,\{(u_1,2v_1),\ldots,(u_m,2v_m)\}} := \left\{ p \in F_{n,2r} : p = \sum_{k=1}^{s} h_{k,1}^{2v_1} \cdots h_{k,m}^{2v_m},\ h_{k,i} \in F_{n,u_i} \right\}. \tag{1.5}$$

There has been recent interest in the cones of convex forms:

$$K_{n,2r} := \{ p \in F_{n,2r} : p \text{ is convex} \}. \tag{1.6}$$

We shall use the two equivalent definitions of "convex" (see e.g. [25, Thm.4.1,4.5]): under the *line segment* definition, p is convex if for all $u, v \in \mathbb{R}^n$ and $\lambda \in [0,1]$,

$$p(\lambda u + (1-\lambda)v) \le \lambda p(u) + (1-\lambda)p(v). \tag{1.7}$$

The *Hessian* definition says that if

$$Hes(p; u, v) := \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 p}{\partial x_i \partial x_j}(u) v_i v_j, \tag{1.8}$$

then p is convex provided $Hes(p; u, v) \ge 0$ for all $u, v \in \mathbb{R}^n$. The cone $K_{n,m}$ appeared in [19], but as $N_{n,m}$ (see Corollary 4.5). Pablo Parrilo asked whether every convex form is sos; that is, is $K_{n,2r} \subseteq \Sigma_{n,2r}$? This question has been answered by Greg Blekherman [3] in the negative. For fixed n, the "probability" that a convex form is sos goes to 0 as $r \to \infty$. No examples of $p \in K_{n,2r} \setminus \Sigma_{n,2r}$ are yet known.

We now give the formal definition of blender. Suppose $n \ge 1$ and $d \ge 0$. The index set for monomials in $F_{n,d}$ consists of n-tuples of non-negative integers:

$$\mathcal{I}(n,d) = \left\{ i = (i_1, \ldots, i_n) : \sum_{k=1}^{n} i_k = d \right\}. \tag{1.9}$$

Write $N(n,d) = \binom{n+d-1}{n-1} = |\mathcal{I}(n,d)|$ and for $i \in \mathcal{I}(n,d)$, let $c(i) = \frac{d!}{i_1! \cdots i_n!}$ be the associated multinomial coefficient. The abbreviation u^i means $u_1^{i_1} \ldots u_n^{i_n}$, where u may be an n-tuple of constants or variables. Every $p \in F_{n,d}$ can be written as

$$p(x_1, \ldots, x_n) = \sum_{i \in \mathcal{I}(n,d)} c(i) a(p; i) x^i. \tag{1.10}$$

The identification of p with the $N(n,d)$-tuple $(a(p;i))$ shows that $F_{n,d} \approx \mathbb{R}^{N(n,d)}$ as a vector space. The topology placed on $F_{n,d}$ is the usual one: $p_m \to p$ means that for every $i \in \mathcal{I}(n,d)$, $a(p_m;i) \to a(p;i)$.

For $\alpha \in \mathbb{R}^n$, define $(\alpha\cdot)^d \in F_{n,d}$ by

$$(\alpha\cdot)^d(x) = \left(\sum_{k=1}^{n} \alpha_k x_k\right)^d = \sum_{i \in \mathcal{I}(n,d)} c(i)\alpha^i x^i. \tag{1.11}$$

If α is regarded as a row vector and x as a column vector, then $(\alpha\cdot)^d(x) = (\alpha x)^d$. If $M = [m_{ij}] \in Mat_n(\mathbb{R})$ is a (not necessarily invertible) real $n \times n$ matrix and $p \in F_{n,d}$, we define $p \circ M \in F_{n,d}$ by

$$(p \circ M)(x_1, \dots, x_n) = p(\ell_1, \dots, \ell_n), \qquad \ell_j(x_1, \dots, x_n) = \sum_{k=1}^{n} m_{jk} x_k. \tag{1.12}$$

If x is viewed as a column vector, then $(p \circ M)(x) = p(Mx)$; $(\alpha\cdot)^d \circ M = (\alpha M\cdot)^d$.

Define $[[p]]$ to be $\{p \circ M : M \in Mat_n(\mathbb{R})\}$, the *closed orbit of* p. If $p = q \circ M$ for *invertible* M, we write $p \sim q$; $\sim$ is an equivalence relation.

Lemma 1.1.
(i) If $p \in F_{n,d}$ and d is odd, then $p \sim \lambda p$ for every $0 \neq \lambda \in \mathbb{R}$.
(ii) If $p \in F_{n,d}$ and d is even, then $p \sim \lambda p$ for every $0 < \lambda \in \mathbb{R}$.
(iii) If $u, \alpha \in \mathbb{R}^n$, then there exists a (singular) M so that $p \circ M = p(u)(\alpha\cdot)^d$.

Proof. For (i), (ii), observe that $(p \circ (cI_n)) = c^d p$ since p is homogeneous, and cI_n is invertible if $c \neq 0$. For (iii), note that if $m_{jk} = u_j \alpha_k$ for $1 \le j, k \le n$, then

$$\ell_j(x) = u_j(\alpha x) \implies (p \circ M)(x_1, \dots, x_n) = (\alpha x)^d p(u_1, \dots, u_n) \tag{1.13}$$

by homogeneity. □

Definition. A set $B \subseteq F_{n,d}$ is a *blender* if these conditions hold:

(P1) If $p, q \in B$, then $p + q \in B$.
(P2) If $p_m \in B$ and $p_m \to p$, then $p \in B$.
(P3) If $p \in B$ and $M \in Mat_n(\mathbb{R})$, then $p \circ M \in B$.

Thus, a blender is a closed convex cone of forms which is also a union of closed orbits. Lemma 1.1 makes it unnecessary to specify in (P1) that $p \in B$ and $\lambda \ge 0$ imply $\lambda p \in B$. Let $\mathcal{B}_{n,d}$ denote the set of blenders in $F_{n,d}$. Trivially, $\{0\}, F_{n,d} \in \mathcal{B}_{n,d}$.

It is simple to see that $P_{n,2r}$ is a blender: conditions (P1) and (P2) can be verified pointwise and if $p(u) \ge 0$ for every u, then the same will be true for $p(Mu)$. Similarly, $K_{n,2r}$ is a blender because (P1) and (P2) follow from the Hessian definition and (P3) follows from the line segment definition.

If $B_1, B_2 \in \mathcal{B}_{n,d}$, then $B_1 \cap B_2 \in \mathcal{B}_{n,d}$. Define the *Minkowski sum*

$$B_1 + B_2 := \{p_1 + p_2 : p_i \in B_i\}. \tag{1.14}$$

The smallest blender containing both B_1 and B_2 must include B_1+B_2; this set is a blender (Theorem 3.4(i)), but it requires an argument to prove (P2). It is not hard to see that $\mathcal{B}_{n,d}$ is not always a chain. Let $(n,d)=(2,8)$ and let $B_1 = W_{2,\{(1,6),(1,2)\}}$ and $B_2 = W_{2,\{(1,4),(1,4)\}}$. Then $x^6y^2 \in B_1$ and $x^4y^4 \in B_2$. If $x^6y^2 \in B_2$, then

$$x^6y^2 = \sum_{k=1}^{s} (\alpha_k x + \beta_k y)^4 (\gamma_k x + \delta_k y)^4. \tag{1.15}$$

The coefficients of x^8 and y^8 show that $\alpha_k\gamma_k = \beta_k\delta_k = 0$ for all k, hence the only non-zero summands are positive multiples of x^4y^4. Thus $x^6y^2 \notin B_2$, and, similarly, $x^4y^4 \notin B_1$, so $B_1 \setminus B_2$ and $B_2 \setminus B_1$ are both non-empty. We do not know simple descriptions of $B_1 \cap B_2$ and $B_1 + B_2$. If $B_1 \in \mathcal{B}_{n,d_1}$ and $B_2 \in \mathcal{B}_{n,d_2}$, define

$$B_1 * B_2 := \left\{ \sum_{k=1}^{s} p_{1,k}p_{2,k} : p_{i,k} \in B_i \right\}. \tag{1.16}$$

Again, this is a blender (Theorem 3.4(ii)), but (P2) is not trivial to prove.

We review some standard facts about convex cones; see [19, Ch.2,3] and [25]. If $C \subset \mathbb{R}^N$ is a closed convex cone, then $u \in C$ is *extremal* if $u = v_1 + v_2, v_i \in C$, implies that $v_i = \lambda_i u$, $\lambda_i \ge 0$. The set of extremal elements in C is denoted $\mathcal{E}(C)$. All cones $C \ne 0, \mathbb{R}^N$ in this paper have the property that $x, -x \in C$ implies $x = 0$. In such a cone, every element in C is a sum of extremal elements. (It will follow from Prop. 2.4 that if $B \in \mathcal{B}_{n,d}$ and $p, -p \in B$ for some $p \ne 0$, then $B = F_{n,d}$.)

As usual, u is *interior* to C if C contains a non-empty open ball centered at u. The set of interior points of C is denoted $int(C)$, and the boundary of C is denoted $\partial(C)$. The next definition depends on the choice of inner product for $\mathbb{R}^N$. For a closed conves cone C, we define the *dual* cone

$$C^* = \{v \in \mathbb{R}^N : [u,v] \ge 0 \quad \text{for all} \quad u \in C\}. \tag{1.17}$$

Then $C^* \subset \mathbb{R}^N$ is also a closed convex cone and $(C^*)^* = C$.

If $u \in C$ (and $\pm x \in C$ implies $x = 0$), then $u \in int(C)$ if and only if $[u,v] > 0$ for every $0 \ne v \in C^*$ (see e.g. [19, p.26]). Thus, if $u \in \partial(C)$ (in particular, if u is extremal), then there exists $v \in C^*$, $v \ne 0$ so that $[u,v] = 0$.

This discussion applies to blenders by identifying $p \in F_{n,d}$ with the $N(n,d)$-tuple of its coefficients. For example, $p \in int(B)$ if there exists $\epsilon > 0$ so that if $|a(q;i)| < \epsilon$ for all $i \in \mathcal{I}(n,d)$, then $p + q \in B$. If $p \sim q \in B$, then p and q simultaneously belong to (or do not belong to) $int(B), \partial(B), \mathcal{E}(B)$. We shall discuss in section two the natural inner product on $F_{n,d}$. It turns out that, under this inner product, $P_{n,2r}$ and $Q_{n,2r}$ are dual cones (Prop. 3.7), as are $K_{n,2r}$ and $W_{n,\{(1,2r-2),(1,2)\}}$ (Theorem 3.10).

The description of $\mathcal{E}(P_{n,2r})$ is extremely difficult if $n \ge 3$. (See e.g [4, 5, 7, 8, 12, 18, 24].) Every element of $\mathcal{E}(\Sigma_{n,2r})$ obviously has the form h^2, but not every square is extremal; e.g.,

$$(x^2+y^2)^2 = (x^2-y^2)^2+(2xy)^2 = \tfrac{1}{18}\left((\sqrt{3}\,x+y)^4 + (\sqrt{3}\,x-y)^4 + 16y^4\right). \tag{1.18}$$

We now describe the contents of this paper. Section two reviews the relevant material from [19] regarding the inner product and its many properties. The principal results are that if $B \in \mathcal{B}_{n,d}$ and $B \neq \{0\}, F_{n,d}$, then $d = 2r$ is even and $Q_{n,2r} \subset \pm B \subset P_{n,2r}$ (Prop. 2.5); the dual cone to a blender is also a blender (Prop. 2.7). Section three begins with a number of preparatory lemmas, mainly involving convergence. We show that if B_i are blenders, then so are $B_1 + B_2$ and $B_1 * B_2$ (Theorem 3.4) and hence the Waring blenders and their generalizations are blenders (Theorems 3.5, 3.6). We show that $P_{n,2r}$ and $Q_{n,2r}$ are dual and give a description of $W^*_{n,(u,2v)}$ (both from [19]) and show that $K_{n,2r}$ and $W_{n,\{(1,2r-2),(1,2)\}}$ are dual (Theorem 3.10). In section four, we consider $K_{n,2r}$. We show that it cannot be decomposed non-trivially as $B_1 * B_2$ (Corollary 4.2), and that $K_{n,2r} = N_{n,2r}$ (c.f. (1.6), (4.4), Corollary 4.5). We also show that if p is positive definite, then $(\sum x_i^2)^N p$ is convex for sufficiently large N (Theorem 4.6). In section five, we show that (up to $\pm$) $\mathcal{B}_{2,4}$ consists of a one-parameter family of blenders B_τ, $\tau \in [-\frac{1}{3}, 0]$, where $\tau = \inf\{\lambda : x^4 + 6\lambda x^2y^2 + y^4 \in B_\tau\}$, increasing from $Q_{2,4} = B_0$ to $P_{2,4} = B_{-\frac{1}{3}}$, and that $B^*_\tau = B_{U(\tau)}$, where $U(\tau) = -\frac{1+3\tau}{3-3\tau}$ (Theorem 5.7). In section six, we review the results of $K_{2,4}$ and $K_{2,6}$ in [9, 10, 17] by Dmitriev and the author, and give some new examples in $\partial(K_{2,2r})$. The full analysis of $\mathcal{E}(K_{2,2r})$ seems intractable for $r \geq 4$. Finally, in section seven, we look at sums of 4th powers of binary forms. Conjecture 7.1 states that $p \in W_{2,(u,4)}$ if and only if $p = f^2 + g^2$, where $f, g \in P_{2,2u}$. We show that this is true for $u = 1$ and for even symmetric octics p (Theorems 7.3, 7.4). Our classification of even symmetric octics implies that

$$x^8 + \alpha x^4y^4 + y^8 \in W_{2,(2,4)} \iff \alpha \geq -\tfrac{14}{9}. \tag{1.19}$$

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2. The inner product

For p and q in $F_{n,d}$, we define an inner product with deep roots in 19th century algebraic geometry and analysis. Let

$$[p, q] = \sum_{i \in \mathcal{I}(n,d)} c(i)a(p; i)a(q; i). \tag{2.1}$$

This is the usual Euclidean inner product, if $p \leftrightarrow (c(i)^{1/2}a(p;i)) \in \mathbb{R}^N$. The many properties of this inner product (see Props. 2.1, 2.6 and 2.9) strongly suggest that this is the "correct" inner product for $F_{n,d}$. We present without proof the following observations about the inner product.

Proposition 2.1. [19, pp.2,3]
(i) $[p,q] = [q,p]$.
(ii) $j \in \mathcal{I}(n,d) \implies [p,x^j] = a[p;j]$.
(iii) $\alpha \in \mathbb{R}^n \implies [p,(\alpha\cdot)^d] = p(\alpha)$.
(iv) If $p_m \to p$, *then* $[p_m,q] \to [p,q]$ *for every* $q \in F_{n,d}$.
(v) In particular, taking $q = (u\cdot)^d$, $p_m \to p \implies p_m(u) \to p(u)$ *for all* $u \in \mathbb{R}^n$.

The orthogonal complement of a subspace U of $F_{n,d}$,

$$U^\perp = \{v \in F_{n,d} : [u,v] = 0 \quad \text{for all} \quad u \in U\}, \tag{2.2}$$

is also a subspace of $F_{n,d}$ and $(U^\perp)^\perp = U$. The following result is widely-known and has been frequently proved over the last century, see e.g.[19, p.30].

Proposition 2.2. [19, p.93] *Suppose* $S \subset \mathbb{R}^n$ *has non-empty interior. Then* $F_{n,d}$ *is spanned by* $\{(\alpha\cdot)^d : \alpha \in S\}$.

Proof. Let U be the subspace of $F_{n,d}$ spanned by $\{(\alpha\cdot)^d : \alpha \in S\}$ and suppose $q \in U^\perp$. Then $0 = [q,(\alpha\cdot)^d] = q(\alpha)$ for all $\alpha \in S$. Since q is a form which vanishes on an open set, $q = 0$. Thus, $U^\perp = \{0\}$, so $U = (U^\perp)^\perp = \{0\}^\perp = F_{n,d}$. □

Proposition 2.3 (Biermann's Theorem). [19, p.31] *The set* $\{(i\cdot)^d : i \in \mathcal{I}(n,d)\}$ *is a basis for* $F_{n,d}$.

Proof. It suffices to construct a dual basis $\{g_j : j \in \mathcal{I}(n,d)\} \subset F_{n,d}$ of $N(n,d)$ forms satisfying $[g_j,(i\cdot)^d] = 0$ if $j \neq i$ and $[g_j,(j\cdot)^d] > 0$. Let

$$g_j(x_1,\dots,x_n) = \prod_{k=1}^{n} \prod_{\ell=0}^{j_k-1} (dx_k - \ell(x_1 + \cdots + x_n)). \tag{2.3}$$

Each g_j is a product of $\sum_k j_k = d$ linear factors, so $g_j \in F_{n,d}$. The (k,ℓ) factor in (2.3) vanishes at any $x = i \in \mathcal{I}(n,d)$ for which $i_k = \ell$. Thus, $[g_j,(i\cdot)^d] = g_j(i) = 0$ if $i_k \le j_k - 1$ for any k. Since $\sum_k i_k = \sum_k j_k$, it follows that $g_j(i) = 0$ if $j \neq i$. A computation shows that $g_j(j) = d^d \prod_k (j_k!) = d^d d!/c(j)$. □

Prop. 2.3 implies Prop. 2.2 directly, by finding an affine copy of $\mathcal{I}(n,d)$ in S.

Proposition 2.4. [19, p.141] *Suppose* $B \in \mathcal{B}_{n,d}$ *and there are forms* $p,q \in B$ *and points* $u,v \in \mathbb{R}^n$ *so that* $p(u) > 0 > q(v)$. *Then* $B = F_{n,d}$.

Proof. By Lemma 1.1(iii), $\pm(\alpha\cdot)^d \in B$ for $\alpha \in \mathbb{R}^n$, so by Prop. 2.2, $F_{n,d} \subseteq B$. □

This is the argument Ellison used in [11, p.667] to show that every form in $F_{n,u(2v+1)}$ is a sum of $(2v+1)$-st powers of forms of degree u.

For $B \in \mathcal{B}_{n,d}$, let $-B = \{-h : h \in B\}$; it is easy to check that $-B \in \mathcal{B}_{n,d}$. Since $Q_{n,2} = P_{n,2}$, the following proposition shows that there are no "interesting" blenders of quadratic forms.

Proposition 2.5. [19, p.141] *If $B \neq \{0\}, F_{n,d}$ is a blender, then $d = 2r$ is even and for a suitable choice of sign, $Q_{n,2r} \subseteq \pm B \subseteq P_{n,2r}$.*

Proof. If $B \neq \{0\}$, then there exists $p \in B$ and $a \in \mathbb{R}^n$ so that $p(a) \neq 0$. If d is odd, then $p(-a) = -p(a)$, and by Prop. 2.4, $B = F_{n,d}$. If d is even, by taking $-B$ if necessary, we may assume that $p(a) \geq 0$. Thus, if $B \neq F_{n,2r}$, then $\pm B \subseteq P_{n,2r}$. On the other hand, Lemma 1.1 and (P1) imply that $Q_{n,2r} \subseteq \pm B$. □

The inner product has a useful contravariant property.

Proposition 2.6. [19, p.32] *Suppose p, $q \in F_{n,d}$ and $M \in Mat_n(\mathbb{R})$. Then*

$$[p \circ M, q] = [p, q \circ M^t]. \tag{2.4}$$

Proof. By Prop. 2.2, it suffices to prove (2.4) for d-th powers; note that $[p \circ M, q] = [(\alpha M \cdot)^d, (\beta \cdot)^d] = (\alpha M \beta^t)^d = (\alpha(\beta M^t)^t)^d = [(\alpha \cdot)^d, (\beta M^t \cdot)^d] = [p, q \circ M^t]$. □

Proposition 2.7. [19, p.46] *If B is a blender, then so is its dual cone B^*.*

Proof. The dual of a closed convex cone is a closed convex cone, so (P1) and (P2) are clear. Suppose $p \in B, q \in B^*$ and $M \in Mat_n(\mathbb{R})$. Since $p \circ M^t \in B$, we have

$$[p, q \circ M] = [q \circ M, p] = [q, p \circ M^t] = [p \circ M^t, q] \geq 0, \tag{2.5}$$

and so $q \circ M \in B^*$. This verifies (P3). □

For $i \in \mathcal{I}(n,d)$, let $D^i = \prod (\frac{\partial}{\partial x_k})^{i_k}$; let $f(D) = \sum c(i) a(f;i) D^i$ be the d-th order differential operator associated to $f \in F_{n,d}$. Since $\frac{\partial}{\partial x_k}$ and $\frac{\partial}{\partial x_\ell}$ commute, $D^i D^j = D^{i+j} = D^j D^i$ for any $i \in \mathcal{I}(n,d)$ and $j \in \mathcal{I}(n,e)$. By multilinearity, $(fg)(D) = f(D)g(D) = g(D)f(D)$ for forms f and g of any degree.

Proposition 2.8. [22, p.183] *If $i, j \in \mathcal{I}(n,d)$ and $i \neq j$, then $D^i(x^j) = 0$ and $D^i x^i = \prod_k (i_k)! = d!/c(i)$.*

Proof. We have

$$D^i(x^j) = \prod_{k=1}^{n} \left(\frac{\partial^{i_k}}{\partial x_k^{i_k}} \right) \prod_{k=1}^{n} x_k^{j_k} = \prod_{k=1}^{n} \frac{\partial^{i_k}(x_k^{j_k})}{\partial x_k^{i_k}}. \tag{2.6}$$

If $i_k > j_k$, then the k-th factor above is zero. If $i \neq j$, then this will happen for at least one k. Otherwise, $i = j$, and the k-th factor is $i_k!$. □

We now connect the inner product with differential operators.

Proposition 2.9. [22, p.184]
(i) If $p, q \in F_{n,d}$, then $p(D)q = q(D)p = d![p,q]$.
(ii) If $p, hf \in F_{n,d}$, where $f \in F_{n,k}$ and $h \in F_{n,d-k}$, then

$$d![p, hf] = (d-k)![h, f(D)p]. \tag{2.7}$$

Proof. For (i), we have by Prop. 2.8:

$$p(D)q = \sum_{i\in\mathcal{I}(n,d)} c(i)a(p;i)D^i\Bigg(\sum_{j\in\mathcal{I}(n,d)} c(j)a(q;j)x^j\Bigg) =$$

$$\sum_{i\in\mathcal{I}(n,d)}\sum_{j\in\mathcal{I}(n,d)} c(i)c(j)a(p;i)a(q;j)D^ix^j = \sum_{i\in\mathcal{I}(n,d)} c(i)c(i)a(p;i)a(q;i)D^ix^i \tag{2.8}$$

$$= \sum_{i\in\mathcal{I}(n,d)} c(i)^2a(p;i)a(q;i)\frac{d!}{c(i)} = d![p,q] = d![q,p] = q(D)p.$$

(ii) Two applications of (i) give

$$d![p,hf] = (hf)(D)p = h(D)f(D)p = h(D)(f(D)p) = (d-k)![h,f(D)p]. \tag{2.9}$$

□

Corollary 2.10. *If $p \in F_{n,2r}$, then $Hes(p;u,v) = 2r(2r-1)[p,(u\cdot)^{2r-2}(v\cdot)^2]$.*

Proof. Apply Prop. 2.9 with $h = (u\cdot)^{2r-2}$, $f = (v\cdot)^2$, $d = 2r$ and $k = 2$. We have

$$f(x_1,\dots,x_n) = (v_1x_1 + \cdots + v_nx_n)^2 \implies f(D) = \sum_{i=1}^n\sum_{j=1}^n v_iv_j\frac{\partial^2}{\partial x_i\partial x_j}, \tag{2.10}$$

so that $[h,f(D)p] = Hes(p;u,v)$ by (1.8) and Prop. 2.1(iii). □

3. Convergence and duality

Throughout this section S will denote the (solid) unit ball in $\mathbb{R}^n$. (The referee generously suggested a more general and much more geometric approach to the results of the first part of this section, using the fact that if C is a compact convex set not containing 0, then the conical hull of C is closed, and a consideration of the behavior of bases of convex cones under Cartesian products.)

Lemma 3.1. *For $i \in \mathcal{I}(n,d)$, there eiststs $R_{n,d}(i) > 0$ so that if $p \in F_{n,d}$, then $|a(p;i)| \le R_{n,d}(i)\cdot\sup\{|p(x)| : x \in S\}$.*

Proof. By Prop. 2.2, there exist $\alpha_k \in S$, so that for every $i \in \mathcal{I}(n,d)$, we have

$$x^i = \sum_{k=1}^{N(n,d)} \lambda_k(i)(\alpha_k\cdot)^d \tag{3.1}$$

for some $\lambda_k(i) \in \mathbb{R}$. Taking the inner product of (3.1) with p, we find that

$$a(p;i) = [p,x^i] = \sum_{k=1}^{N(n,d)} \lambda_k(i)[p,(\alpha_k\cdot)^d] = \sum_{k=1}^{N(n,d)} \lambda_k(i)p(\alpha_k). \tag{3.2}$$

Now set $R_{n,d}(i) = \sum_k |\lambda_k(i)|$. □

We define a norm on $F_{n,d}$ by

$$||p||^2 = p(D)p = d![p,p] = d! \sum_{i \in \mathcal{I}(n,d)|} c(i)a(p;i)^2. \tag{3.3}$$

This norm satisfies a remarkable inequality due to Beauzamy, Bombieri, Enflo and Montgomery [1] (see [20] for this formulation): if $p \in F_{n,d_1}$ and $q \in F_{n,d_2}$, then

$$||pq|| \ \geq ||p|| \cdot ||q||. \tag{3.4}$$

Given a sequence $(p_m) \in F_{n,d}$, the statement that $(|a(p_m;i)|)$ is uniformly bounded for all (i,m) is equivalent to the statement that $(||p_m||)$ is bounded.

Lemma 3.2. *Suppose $(p_{m,r}) \subset F_{n,d}$, $1 \leq r \leq N$, and suppose that for all (m,r), $|p_{m,r}(u)| \leq M$ for $u \in S$. Then there exist $p_r \in F_{n,d}$ and a common subsequence $m_k \to \infty$ so that $p_{m_k,r} \to p_r$ for each r.*

Proof. Identify $p_{m,r}$ with the vector $(a(p_{m,r};i)) \in \mathbb{R}^{N(n,d)}$; these are uniformly bounded by Lemma 3.1. Concatenate them to form a vector $v_m \in \mathbb{R}^{N*N(n,d)}$. By Bolzano-Weierstrass, there is a convergent subsequence (v_{m_k}). The corresponding subsequences of forms are then convergent. □

We state without proof a direct implementation of Carathéodory's Theorem (see e.g. [19, p.27].). It is worth noting that in 1888 (when Carathéodory was 15), Hilbert [13] used this argument with $N(3,6) = 28$ to show that $\Sigma_{3,6}$ is closed.

Proposition 3.3 (Carathéodory's Theorem). *If $r > N(n,d)$, and $h_k \in F_{n,d}$, then there exist $\lambda_k \geq 0$ so that*

$$\sum_{k=1}^{r} h_k = \sum_{k=1}^{N(n,d)} \lambda_k h_{n_k}. \tag{3.5}$$

We use these lemmas to show that if B_1 and B_2 are blenders, then so are $B_1 + B_2$ (c.f. (1.14)) and $B_1 * B_2$ (c.f. (1.16)). We may assume $B_i \neq 0$.

Theorem 3.4.

(i) If $B_i \in \mathcal{B}_{n,2r}$, then $B_1 + B_2 \in \mathcal{B}_{n,2r}$.
(ii) If $B_i \in \mathcal{B}_{n,2r_i}$ and $r = r_1 + r_2$, then $B_1 * B_2 \in \mathcal{B}_{n,2r}$.

Proof. In each case, (P1) is automatic, and since $(p_1 + p_2) \circ M = p_1 \circ M + p_2 \circ M$ and $(p_1p_2) \circ M = (p_1 \circ M)(p_2 \circ M)$, (P3) is verified. The issue is (P2).

Suppose $B_i \in \mathcal{B}_{n,2r}$ have opposite "sign", say $B_1 \subset P_{n,2r}$ and $B_2 \subset -P_{n,2r}$. Then Prop. 2.4 implies that $B_1 + B_2 = F_{n,2r}$. Otherwise, we may assume that $B_i \subset P_{n,2r_i}$. Suppose $p_{i,m} \in B_i$ and $p_{1,m} + p_{2,m} = p_m \to p$. If $\sup\{p(u) : u \in S\} = T$, then for $m \geq m_0$, $\sup\{p_m(u) : u \in S\} \leq T + 1$, and since $p_{i,m}$ is psd, it follows that $\sup\{p_{i,m}(u) : u \in S\} \leq T + 1$ as well. By Lemma 3.2, there is a common subsequence so that $p_{i,m_k} \to p_i \in B_i$, hence $p = \lim p_{m_k} = p_1 + p_2 \in B_1 + B_2$.

The proof for products is more complicated; the example $(mp_1)(m^{-1}p_2) = p_1p_2$ shows that the factors might need to be normalized. By taking $\pm B_i$, assume

$B_i \subset P_{n,2r_i}$. Suppose first that $p_{i,m} \in B_i$ and $p_{1,m}p_{2,m} \to p \in P_{n,2r_1+2r_2}$. If $p = 0$, then $p \in B_1 * B_2$. Otherwise, assume that $p_{i,m} \neq 0$. Let $\lambda_m = (||p_{1,m}||/||p_{2,m}||)^{1/2}$, $q_{1,m} = \lambda_m^{-1} p_{1,m}$ and $q_{2,m} = \lambda_m p_{2,m}$. Then $q_{i,m} \in B_i$, $q_{1,m}q_{2,m} \to p$ and $||q_{1,m}|| = ||q_{2,m}||$. It follows from (3.4) that $\limsup ||q_{i,m}|| \le ||p||^{1/2}$, hence the $q_{i,m}$'s have bounded norm and again, there exists m_k so that $q_{i,m_k} \to q_i \in B_i$ and $p = q_1q_2$.

By Prop. 3.3, a sum such as (1.16) can be compressed into one in which $s \le N(n, 2r)$. Write

$$p_m = \sum_{k=1}^{N(n,2r)} p_{1,k,m}p_{2,k,m}, \qquad p_{i,k,m} \in B_i, \tag{3.6}$$

and suppose $p_m \to p$. Since p is bounded on S, so is (p_m), and since each $p_{i,k,m}$ is psd, it follows that the sequence $(p_{1,k,m}p_{2,k,m})$ is bounded on S, and hence by Lemma 3.2, a subsequence of $(p_{1,k,m}p_{2,k,m}) \to p_k$ for some $p_k \in P_{n,2r}$; without loss of generality, we may drop the subscripts as assume that $(p_{1,k,m}p_{2,k,m}) \to p_k$. We now apply the argument of the previous paragraph to complete the proof. □

The following theorem was announced without proof in [19, p.47].

Theorem 3.5. *If $uv = r$, then $W_{n,(u,2v)}$ is a blender.*

Proof. As we have seen, (P1) and (P3) are immediate. Suppose $p_m \in W_{n,(u,2v)}$ and $p_m \to p$. Prop. 3.3 says that we can write

$$p_m = \sum_{k=1}^{N(n,2r)} h_{k,m}^{2v}, \qquad h_{k,m} \in F_{n,u}. \tag{3.7}$$

As before, p is bounded on S, so the p_m's are bounded, hence so are the sequences $(h_{k,m}^{2v})$ and $(|h_{k,m}|) = ((h_{k,m}^{2v})^{1/(2v)})$. Thus, there is a common convergent subsequence so that $(h_{k,m_\ell}) \to h_k$, hence $(h_{k,m_\ell}^{2v}) \to h_k^{2v}$ and $p \in W_{n,(u,2v)}$. □

In particular, $\Sigma_{n,2r}$ and $Q_{n,2r}$ are blenders; see [19, p.46].

Theorem 3.6. *If $\sum_i u_iv_i = 2r$, then $W_{n,\{(u_1,2v_1),\dots,(u_m,2v_m)\}} \in \mathcal{B}_{n,2r}$.*

Proof. Note that $W_{n,\{(u_1,2v_1),\dots,(u_m,2v_m)\}} = W_{n,(u_1,2v_1)} * \cdots * W_{n,(u_m,2v_m)}$. □

Proposition 3.7. [19, p.38] *$P_{n,2r}$ and $Q_{n,2r}$ are dual blenders.*

Proof. We have $p \in Q_{n,2r}^*$ if and only if $p \in F_{n,2r}$ and $\lambda_k \ge 0$ and $\alpha_k \in \mathbb{R}^n$ imply

$$0 \le \left[p, \sum_{k=1}^{r} \lambda_k (\alpha_k \cdot)^{2r} \right] = \sum_{k=1}^{r} \lambda_k p(\alpha_k). \tag{3.8}$$

This holds if and only if $p(\alpha) \ge 0$ for $\alpha \in \mathbb{R}^n$; that is, if and only if $p \in P_{n,2r}$. □

It was a commonplace by the time of [13] that $P_{n,2r} = \Sigma_{n,2r}$ when $n = 2$ or $2r = 2$. Hilbert proved there that $P_{3,4} = \Sigma_{3,4}$ and that strict inclusion is true for other $(n, 2r)$ (see [23].) We say that $p \in P_{n,2r}$ is *positive definite* or *pd* if $p(u) = 0$ only for $u = 0$. It follows that $p \in int(P_{n,2r})$ if and only if p is pd.

Blenders are cousins of orbitopes. An *orbitope* is the convex hull of an orbit of a compact algebraic group G acting linearly on a real vector space; see [26, p.1]. The key differences from blenders are that it is a single orbit, and that G is compact. One object which is both a blender and an orbitope is $Q_{n,2r}$, which is named $\mathcal{V}_{n,2r}$ (and called the *Veronese orbitope*) in [26].

Proposition 3.8. [19, p.47] *Given $p \in F_{n,2uv}$, define the form $H_p(t) \in F_{N(n,u),2v}$, in variables $\{t(\ell)\}$ indexed by $\{\ell \in \mathcal{I}(n,u)\}$, by*

$$H_p(\{t(\ell_j)\}) = \sum_{\ell_1 \in \mathcal{I}(n,u)} \cdots \sum_{\ell_{2v} \in \mathcal{I}(n,u)} a(p; \ell_1 + \cdots + \ell_{2v}) t(\ell_1) \cdots t(\ell_{2v}). \tag{3.9}$$

Then $p \in W^*_{n,(u,2v)}$ if and only if $H_p \in P_{N(n,u),2v}$.

Proof. We have $p \in W^*_{n,(u,v)}$ if and only if, for every form $g \in F_{n,u}$, $[p, g^{2v}] \geq 0$. Writing $g \in F_{n,u}$ with coefficients $\{t(\ell) : \ell \in \mathcal{I}(n,u)\}$, we have:

$$\begin{aligned} g(x) &= \sum_{\ell \in \mathcal{I}(n,u)} t(\ell) x^{\ell} \implies \\ g^{2v}(x) &= \sum_{\ell_1 \in \mathcal{I}(n,u)} \cdots \sum_{\ell_{2v} \in \mathcal{I}(n,u)} t(\ell_1) \cdots t(\ell_{2v}) x^{\ell_1 + \cdots + \ell_{2v}}. \end{aligned} \tag{3.10}$$

It follows from (2.1) and (3.9) that $[p, g^{2v}] = H_p(t(\ell))$. □

If $v = 1$, then $\mathcal{I}(n,1) = \{e_i\}$ and, on writing $t(e_i) = y_i$, $H_p(y_1, \dots, y_n) = p(y)$; i.e., $Q^*_{n,2r} = P_{n,2r}$. If $u = 1$, then H_p becomes the classical catalecticant and

$$p \in \Sigma^*_{2,2r} \iff H_p(t) = \sum_{i \in \mathcal{I}(n,r)} \sum_{j \in \mathcal{I}(n,r)} a(p; i + j) t(\ell_i) t(\ell_j) \quad \text{is } psd. \tag{3.11}$$

This shows that $\Sigma_{n,2r}$ is a spectrahedron (see [26, p.27]).

Theorem 3.9. *If $\sum v_i = r$, then $W_{2,\{(1,2v_1),\dots,(1,2v_m)\}} = P_{2,2r}$ if and only if $m = r$ and $v_i = 1$.*

Proof. If $p \in P_{2,2r} = \Sigma_{2,2r}$, then $p = f_1^2 + f_2^2$, where $f_i \in F_{2,r}$. Factor $\pm f_i$ into a product of linear and pd quadratic factors (themselves a sum of two squares):

$$f_i = \prod_j \ell_{1,j} \prod_k (\ell_{2,k}^2 + \ell_{3,k}^2). \tag{3.12}$$

Then, using (1.18) and expanding the product below, we see that

$$f_i^2 = \prod_j \ell_{1,j}^2 \prod_k \big((\ell_{2,k}^2 - \ell_{3,k}^2)^2 + (2\ell_{2,k}\ell_{3,k})^2 \big) \in W_{2,\{(1,2),\dots,(1,2)\}}. \tag{3.13}$$

The converse inclusion follows from Prop. 2.5.

Suppose $m < r$ and suppose

$$\prod_{\ell=1}^{r} (x - \ell y)^2 = \sum_{k=1}^{s} h_{k,1}^{2v_1} \cdots h_{k,m}^{2v_m}, \quad h_{k,i}(x,y) = \alpha_{k,i} x + \beta_{k,i} y \in F_{2,1}. \tag{3.14}$$

Then for each k, we have

$$\prod_{\ell=1}^{r}(x-\ell y) \;\Big|\; \prod_{i=1}^{m}(\alpha_{k,i}x+\beta_{k,i}y); \tag{3.15}$$

since $m<r$, the right-hand side is 0, and we have a contradiction. □

Finally, we have a simple expression for $K_{n,2r}^*$ which is implicit in [3].

Theorem 3.10. *$K_{n,2r}$ and $W_{n,\{(1,2r-2),(1,2)\}}$ are dual blenders.*

Proof. By Corollary 2.10 and the Hessian definition, p is convex if and only if $0 \le Hes(p;u,v) = 2r(2r-1)[p,(u\cdot)^{2r-2}(v\cdot)^2]$ for all $u,v\in\mathbb{R}^n$. □

It follows from Theorems 3.9 and 3.10 that $K_{2,4}^* = W_{2,\{(1,2),(1,2)\}} = P_{2,4}$, so $K_{2,4}=Q_{2,4}$. For $r\ge 3$, $K_{2,2r}^* = W_{2,\{(1,2r-2),(1,2)\}} \subsetneq P_{2,4}$, so $K_{2,2r} \supsetneq Q_{2,2r}$. We return to this topic in section six.

4. $K_{n,2r}$: convex forms

In this section, we prove some general results for $K_{n,2r}$. Since $p\in K_{n,2r}$ if and only if $Hes(p;u,v)$ is psd and $Hes(p;u,u)=2r(2r-1)p(u)$, we get an alternative proof that $K_{2,2r}\subseteq P_{n,2r}$. We also know from Theorem 3.10 that $p\in int(K_{n,2r})$ if and only if $[p,q]>0$ for $0\ne q\in W_{n,\{(1,2r-2),(1,2)\}}$; accordingly, $int(K_{n,2r})$ is the set of $p\in K_{2,2r}$ so that $Hes(p;u,v)$ is positive definite as a bihomogeneous form in the variables $u\in\mathbb{R}^n$ and $v\in\mathbb{R}^n$. Equivalently, $p\in K_{n.2r}$ is in $\partial(K_{n,2r})$ if and only if there exist $u_0\ne 0, v_0\ne 0$ such that $Hes(p;u_0,v_0)=0$.

Although psd and sos are preserved under homogenization and dehomogenization, this is not true for convexity. For example, t^2-1 is a convex polynomial which cannot be homogenized to a convex form, because it is not definite. As a pd polynomial in one variable, t^4+12t^2+1 is convex, but if $p(x,y)=x^4+12x^2y^2+y^4$, then $Hes(p;(1,1),(v_1,v_2)) = 36v_1^2+96v_1v_2+36v_2^2$ is not psd, so p is not convex.

Proposition 4.1. *If $p\in K_{n,2r}$, then there is a pd form q in $\le n$ variables and $\bar{p}\sim p$ such that $\bar{p}(x)=q(x_k,\dots,x_n)$.*

Proof. If p is pd, there is nothing to prove. Otherwise, we can assume that $p\sim\bar{p}$, where $\bar{p}$ is convex and $\bar{p}(e_1)=0$. We shall show that $\bar{p}=\bar{p}(x_2,\dots,x_n)$. Repeated application of this argument then proves the result.

Suppose otherwise that x_1 appears in a term of $\bar{p}$ and let $m\ge 1$ be the largest such power of x_1; write the associated terms in $\bar{p}$ as $x_1^m h(x_2,...,x_n)$. After an additional invertible linear change involving $(x_2,\dots,x_n)$, we may assume that one of these terms is $x_1^m x_2^{2r-m}$. We then have

$$\bar{p}(x_1,x_2,0,\dots,0) = x_1^m x_2^{2r-m} + \text{lower order terms in } x_1 \tag{4.1}$$

which implies that

$$\frac{\partial^2\bar{p}}{\partial x_1^2}\frac{\partial^2\bar{p}}{\partial x_2^2} - \left(\frac{\partial^2\bar{p}}{\partial x_1\partial x_2}\right)^2 = \\ -(2r-1)m(2r-m)x_1^{2m-2}x_2^{4r-2m-2} + \text{lower order terms in } x_1. \tag{4.2}$$

Since $r \ge 1$ and $1 \le m \le 2r-1$, (4.2) cannot be psd, and this contradiction shows that x_1 does not occur in $\bar{p}$. □

Corollary 4.2. *There do not exist $B_i \in \mathcal{B}_{n,2r_i}$, $r_i \ge 1$, so that $K_{n,2r_1+2r_2} = B_1 * B_2$.*

Proof. It follows from Prop. 2.5 that $x_i^{2r_i} \in B_i$, hence $x_1^{2r_1}x_2^{2r_2} \in B_1 * B_2$, but by Prop. 4.1, this form is not convex. □

The next theorem connects $K_{n,2r}$ with the blender $N_{n,2r}$ defined in [19, p.119-120]. Let $E = <e_1,\dots,e_n>$ be a real n-dimensional vector space. We say that f is a *norm-function* on E if, after defining

$$||x_1e_1 + \cdots + x_ne_n|| = f(x_1,\dots,x_n), \tag{4.3}$$

the pair $(E, ||\cdot||)$ is a Banach space. Let

$$N_{n,d} := \{p \in F_{n,d} : p^{1/d} \text{ is a norm function}\}. \tag{4.4}$$

A necessary condition is that $f = p^{1/d} \ge 0$, hence $d = 2r$ is even and $p \in P_{n,2r}$. For example, if $p(x) = \sum_k x_k^2$, then (4.3) with $f = p^{1/2}$ gives $\mathbb{R}^n$ with the Euclidean norm. If $(E, ||\cdot||)$ is isometric to a subspace of some $L_{2r}(X,\mu)$, then $f^{2r} \in Q_{n,2r}$. The following theorem was proved in the author's thesis; see [16, 17].

Proposition 4.3. [17, Thm.1] *If $p \in P_{n,2r}$, then $p \in N_{n,2r}$ if and only if for all $u, v \in \mathbb{R}^n$, $p(u_1+tv_1,\dots,u_n+tv_n)^{1/(2r)}$ is a convex function of t.*

It is not obvious that $N_{n,2r}$ is a blender, but in fact, $N_{n,2r} = K_{n,2r}$! The connection is a proposition whose provenance is unclear. It appears in Rockafellar [25, Cor.15.3.1], where it is attributed to Lorch [14], although the derivation is not transparent. V. I. Dmitriev (see section 6) attributes the result to an observation by his advisor S. G. Krein in 1969. Note below that q is *not* homogeneous.

Proposition 4.4. *Suppose $p \in P_{n,2r}$ and $p(1,0,...,0) > 0$. Let*

$$q(x_2,\dots,x_n) = p(1,x_2,\dots,x_n). \tag{4.5}$$

Then $p \in K_{n,2r}$ if and only if $q^{1/(2r)}(x_2,\dots,x_n)$ is convex.

Corollary 4.5. $K_{n,2r} = N_{n,2r}$.

Proof of Prop. 4.4. A function is convex if and only if it is convex when restricted to all two-dimensional subspaces. Consider all $a \in \mathbb{R}^N$ with $a_1 = 1$. Suppose we can show that $Hes(p;a,u)$ is psd in u if and only if $q^{1/(2r)}$ is convex at $(a_2,\dots,a_n)$. By homogeneity, this occurs if and only if $Hes(p;a,u)$ is psd in u for every a with $a_1 \ne 0$ and by continuity, this holds if and only if $Hes(p;a,u)$ is psd for all a, u. Thus, it suffices to set $a_1 = 1$ and prove the equivalence pointwise.

Fix $(a_2, \dots, a_n)$ and let

$$\begin{aligned} \tilde{p}(x_1, x_2 \dots, x_n) &= p(x_1, x_2 + a_2 x_1, \dots, x_n + a_n x_1), \\ \tilde{q}(x_2, \dots, x_n) &= \tilde{p}(1, x_2, \dots x_n) = q(x_2 + a_2, \dots, x_n + a_n) \end{aligned} \tag{4.6}$$

Then p and $q^{1/(2r)}$ are convex at a and $(a_2, \dots, a_n)$ if and only if $\tilde{p}$ and $\tilde{q}$ are convex at e_1 and 0, and we can drop the tildes and assume that $a_k = 0$ for $k \ge 2$, so $a = e_1$. Since it suffices to look at all two-dimensional subspaces containing e_1, we may assume it is $\{(x_1, x_2, 0, \dots, 0)\}$, after another change of variables.

Suppose now that

$$h(x_1, x_2) = p(x_1, x_2, 0, \dots, 0) = a_0 x_1^{2r} + \binom{2r}{1} a_1 x_1^{2r-1} x_2 + \dots. \tag{4.7}$$

Then

$$Hes(h; (1,0), (v_1, v_2)) = 2r(2r-1)(a_0 v_1^2 + 2a_1 v_1 v_2 + a_2 v_2^2), \tag{4.8}$$

and since $a_0 = p(e_1) > 0$, this is psd if and only if $a_0 a_2 \ge a_1^2$. On the other hand,

$$q(t) = p(1, t) = a_0 + \binom{2r}{1} a_1 t + \binom{2r}{2} a_2 t^2 + \dots \tag{4.9}$$

and a routine computation shows that

$$(q^{(1/(2r))})''(0) = (2r-1) a_0^{-2+1/(2r)} (a_0 a_2 - a_1^2). \tag{4.10}$$

Thus the two conditions hold simultanously. □

A more complicated proof computes the Hessian of p and uses the Euler PDE ($2rp = \sum x_i \frac{\partial p}{\partial x_i}$ and $(2r-1)\frac{\partial p}{\partial x_i} = \sum x_j \frac{\partial^2 p}{\partial x_i \partial x_j}$) to replace partials involving x_1 with partials involving x_j, $j \ge 2$.

We conclude this section with a peculiar result which implies that every pd form is, in a computable way, the restriction of a convex form on S^{n-1}.

Theorem 4.6. *Suppose $p \in P_{n,2r}$ is pd, and let $p_N := (\sum_j x_j^2)^N p$. Then there exists N so that $p_N \in K_{n,2r+2N}$.*

Proof. Since p is pd, it is bounded away from 0 on S^{n-1} and so there are uniform upper bounds T for $|p(x)^{-1} \nabla_u(p)(x)|$ and U for $|p(x)^{-1} \nabla_u^2(p)(x)|$, for $x, u \in S^{n-1}$. Since $\sum x_i^2$ is rotation-invariant, once again it suffices to show that p_N is convex at $(1, 0, \dots, 0)$, given $x_3 = \dots = x_n = 0$. We claim that if $N > (T^2 + U)/2$, then p_N is convex. By Prop. 4.4, it suffices to show that $p_N^{1/(2N+2r)}(1, t, 0, \dots, 0)$ is convex at $t = 0$. Writing down the relevant Taylor series, this becomes

$$(1 + t^2)^{N/(2N+2r)} (1 + \alpha t + \tfrac{1}{2}\beta t^2 + \dots)^{1/(2N+2r)}, \tag{4.11}$$

where $|\alpha| \le T$ and $|\beta| \le U$. By expanding the product, a standard computation shows that the second derivative at $t = 0$ is

$$\frac{N}{N+r} + \frac{1}{2N+2r} \cdot b - \frac{2N+2r-1}{(2N+2r)^2} \cdot a^2 \ge \frac{1}{2N+2r} \left(2N - U - T^2\right) \ge 0. \tag{4.12}$$

□

Greg Blekherman pointed out to the author's chagrin in Banff that Theorem 4.6 follows from [21, Thm.3.12]: if p is pd, then there exists computable N so that $p_N \in Q_{n,2r+2N}$. This was used in [21] to show that $P_N \in \Sigma_{n,2r+2N}$; it also implies that $p \in K_{n,2r+2N}$. The proof of [21, Thm.3.12] is much less elementary.

We conclude this section with a computational illustration of Theorem 4.6. If $a \ge 0$, then $x^2 + ay^2$ is convex, but if $r \ge 1$ and $(x^2 + y^2)^r(x^2 + ay^2) \in K_{2,2r+2}$ for all $a > 0$, then by (P2), $x^2(x^2 + y^2)^r$ would be convex, violating Prop. 4.1.

Theorem 4.7.

$$(x^2 + y^2)^r(x^2 + ay^2) \in K_{2,2r+2} \iff a + 1/a \le 8r + 18 + 8/r. \tag{4.13}$$

Proof. Let $p(x, y) = (x^2 + y^2)^r(x^2 + ay^2)$. Then $\frac{\partial^2 p}{\partial x^2}\frac{\partial^2 p}{\partial y^2} - (\frac{\partial^2 p}{\partial x \partial y})^2$ equals

$$\begin{gathered} 4(2r+1)(x^2+y^2)^{2r-2}q(x,y), \quad \text{where} \quad q(x,y) = \\ (1+r)(a+r)x^4 + (2a - r + 6ar - a^2r + 2ar^2)x^2y^2 + a(1+r)(1+ar)y^4. \end{gathered} \tag{4.14}$$

Another computation shows that

$$\begin{aligned} &4(1+r)(a+r)q(x,y) \\ &= (2(1+r)(a+r)x^2 + (2a - r + 6ar - a^2r + 2ar^2)y^2)^2 \\ &\quad + ar^2(a-1)^2\big((8r + 18 + 8/r) - (a + 1/a)\big)y^4. \end{aligned} \tag{4.15}$$

If $a + 1/a \le 8r + 18 + 8/r$, then (4.15) shows that q is psd. Suppose $a + 1/a > 8r + 18 + 8/r$. Observe that $2a - r + 6ar - a^2r + 2ar^2 \ge 0$ if and only if $(a + 1/a) \le 2r + 6 + 2/r$, so in this case, $2a - r + 6ar - a^2r + 2ar^2 < 0$ and we can choose $(x, y) = (x_0, y_0) \ne (0, 0)$ to make the first square in (4.15) equal to zero. It then follows that $4(1 + r)(a + r)q(x_0, y_0) < 0$. □

In particular, $(x^2 + y^2)(x^2 + ay^2) \in K_{2,4} \iff 17 - 12\sqrt{2} \le a \le 17 + 12\sqrt{2}$.

5. $\mathcal{B}_{2,4}$: binary quartic blenders

In view of Prop. 2.5, the simplest non-trivial opportunity to classify blenders comes with the binary quartics. Throughout this section, we choose a sign for $\pm B \in \mathcal{B}_{2,4}$ and assume that $B \subset P_{2,4}$. We shall show that $\mathcal{B}_{2,4}$ is a one-parameter nested family of blenders increasing from $Q_{2,4}$ to $P_{2,4}$. Let $Z_{2,4}$ denote the set of $p \in P_{2,4}$ which are neither pd not a 4th power; if $p \in Z_{2,4}$, then $p = \ell^2 h$, where ℓ is linear and h is a psd quadratic form relatively prime to ℓ.

Lemma 5.1. *If $B \in \mathcal{B}_{2,4}$ and $0 \ne p \in B \cap Z_{2,4}$, then $B = P_{2,4}$.*

Proof. We have $p \sim q$, where $q(x, y) = x^2(ax^2 + 2bxy + cy^2) \in B$, $ac - b^2 \ge 0$ and $c > 0$. But

$$x^2(ax^2 + 2bxy + cy^2) = x^2\Big(\big(\tfrac{ac-b^2}{c}\big)x^2 + c\big(\tfrac{b}{c}x + y\big)^2\Big) \sim x^2(dx^2 + cy^2), \tag{5.1}$$

and $d \ge 0$. Next, $(x, y) \mapsto (\epsilon x, \epsilon^{-1}y)$ shows that $\epsilon^2 dx^4 + cx^2y^2 \in B$, so $x^2y^2 \in B$ by (P2) and $\ell_1^2\ell_2^2 \in B$ by (P3). Thus, $W_{2,\{(1,2),(1,2)\}} = P_{2,4} \subseteq B$ by Theorem 3.9. □

This lemma illustrates one difference between blenders and orbitopes. If $G = SO(2)$ and $p(x,y) = x^2(x^2+y^2)$, then the convex hull of the image of p under G will be $\mathrm{cvx}(\{(\cos tx + \sin ty)^2(x^2+y^2)\})$, which contains no 4th powers.

Two important families of binary quartics are:

$$f_\lambda(x,y) := x^4 + 6\lambda x^2y^2 + y^4; \tag{5.2}$$

$$g_\lambda(x,y) := f_\lambda(x+y,x-y) = (2+6\lambda)x^4 + (12-12\lambda)x^2y^2 + (2+6\lambda)y^4. \tag{5.3}$$

We shall need two special fractional linear transformations. Let

$$T(z) := \frac{1-z}{1+3z}, \qquad U(z) := -\frac{1+3z}{3-3z}. \tag{5.4}$$

Thus, $g_\lambda = (2+6\lambda)f_{T(\lambda)}$, hence for $\lambda \neq -\frac{1}{3}$, $f_\lambda \sim f_{T(\lambda)}$. Note that $T(T(z)) = z$, $T(0) = 1$, $T(\frac{1}{3}) = \frac{1}{3}$, and $T(-\frac{1}{3}) = \infty$ (corresponding to $(x^2-y^2)^2 \sim x^2y^2$); T gives a 1-1 decreasing map between $[\frac{1}{3},\infty)$ and $(-\frac{1}{3},\frac{1}{3}]$. A calculation shows that

$$[f_\lambda, g_\mu] = (2+6\mu) + \lambda(12-12\mu) + (2+6\mu) = 4(1+3\lambda+3\mu-3\lambda\mu). \tag{5.5}$$

Note that $U(U(z)) = z$, $U(0) = -\frac{1}{3}$, U gives a 1-1 decreasing map from $[-\frac{1}{3},0]$ to itself, and

$$[f_\lambda, g_{(U(\lambda)+\tau)}] = 12(1-\lambda)\tau. \tag{5.6}$$

It follows from (5.6) that $[f_\lambda, g_{U(\lambda)}] = 0$; if $\lambda < 1$ and $\mu < U(\lambda)$, then $[f_\lambda, g_\mu] < 0$.

It is easy to see directly from (5.2) that f_λ is psd if and only if $\lambda \in [-\frac{1}{3},\infty)$, and pd if and only if $\lambda \in (-\frac{1}{3},\infty)$, and from (P3) that, if $B \in \mathcal{B}_{2,4}$, then

$$f_\lambda \in B \iff f_{T(\lambda)} \in B. \tag{5.7}$$

By (P1), if $-\frac{1}{3} < \lambda \le \frac{1}{3}$, then $f_\lambda \in B$ implies that $f_\mu \in B$ for $\mu \in [\lambda, T(\lambda)]$.

Classically, a "general" real binary quartic can be put into the shape f_λ after an invertible linear transformation. However the coefficients of might not be real, and there are singularities: $x^4 \not\sim f_\lambda$. The following result is [15, Thm.6].

Proposition 5.2. *If $p \in P_{2,4}$ is pd, then $p \sim f_\lambda$ for some $\lambda \in (-\frac{1}{3},\frac{1}{3}]$.*

Proof. Suppose first $p = g^2$. Then g is pd, so $g \sim x^2+y^2$ and $p \sim f_{\frac{1}{3}}$.

If p is not a perfect square, then it is a product of two pd quadratic forms; we may assume that $p(x,y) = (x^2+y^2)q(x,y)$, with

$$q(x,y) = ax^2 + 2bxy + cy^2. \tag{5.8}$$

A "rotation of axes" fixes x^2+y^2 and takes q into dx^2+ey^2 with $d,e>0$, $d \neq e$, so $p \sim (x^2+y^2)(dx^2+ey^2)$. Now, $(x,y) \mapsto (d^{-1/4}x, e^{-1/4}y)$ gives $p \sim f_\mu$, where $\mu = \frac{1}{6}(\gamma+\gamma^{-1}) > \frac{1}{3}$ for $\gamma = \sqrt{d/e} \neq 1$. Thus, $p \sim f_{T(\mu)}$ where $T(\mu) \in (-\frac{1}{3},\frac{1}{3})$. $\square$

We need some results from classical algebraic geometry. Suppose

$$p(x,y) = \sum_{k=0}^{4} \binom{4}{k} a_k(p) x^{4-k} y^k. \tag{5.9}$$

The two "fundamental invariants" of p are

$$I(p) = a_0(p)a_4(p) - 4a_1(p)a_3(p) + 3a_2(p)^2,$$
$$J(p) = \det \begin{vmatrix} a_0(p) & a_1(p) & a_2(p) \\ a_1(p) & a_2(p) & a_3(p) \\ a_2(p) & a_3(p) & a_4(p) \end{vmatrix}. \tag{5.10}$$

(Here, $J(p)$ is the determinant of the catalecticant matrix H_p.) We have $I(f_\lambda) = 1 + 3\lambda^2$ and $J(f_\lambda) = \lambda - \lambda^3$, but $I(x^4) = J(x^4) = 0$. It follows from Prop. 5.2 that if p is pd, then $I(p) > 0$, and, classically, if $q(x,y) = p(ax+by, cx+dy)$, then

$$I(q) = (ad-bc)^4 I(p), \qquad J(q) = (ad-bc)^6 J(p). \tag{5.11}$$

Let

$$K(p) := \frac{J(p)}{I(p)^{3/2}}. \tag{5.12}$$

It follows from (5.11) and (5.12) that, if $p \sim q$, then $K(q) = K(p)$. In particular,

$$p \sim f_\lambda \implies K(p) = K(f_\lambda) = \phi(\lambda) := \frac{\lambda - \lambda^3}{(1+3\lambda^2)^{3/2}}. \tag{5.13}$$

Lemma 5.3. *If p is pd, then $p \sim f_\lambda$, where λ is the unique solution in $(-\frac{1}{3}, \frac{1}{3}]$ to $K(p) = \phi(\lambda)$. If $p \in Z_{2,4}$, then $K(p) = \phi(-\frac{1}{3})$.*

Proof. By Proposition 5.2, $p \sim f_\lambda$ for some $\lambda \in (-\frac{1}{3}, \frac{1}{3}]$. A routine computation shows that $f'(\lambda) = (1-9\lambda^2)(1+3\lambda^2)^{-5/2}$ is positive on $(-\frac{1}{3}, \frac{1}{3})$, hence ϕ is strictly increasing. By Lemma 5.1, if $p \in Z_{2,4}$, then $p \sim q$, where $q(x,y) = dx^4 + 6ex^2y^2$ for some $e > 0$. Since $I(q) = 3e^2$ and $J(q) = -e^3$, $K(p) = K(q) = 3^{-3/2} = \phi(-\frac{1}{3})$. $\square$

Theorem 5.4. *Suppose $r, s \in [-\frac{1}{3}, 0]$, and suppose $1 + 3r + 3s - 3rs = 0$; that is, $s = U(r)$. If $p \in [[f_r]]$ and $q \in [[f_s]]$, then $[p,q] \ge 0$.*

Proof. Suppose $p = f_r \circ M_1$ and $q = f_s \circ M_2$. Then

$$[p,q] = [f_r \circ M_1, f_s \circ M_2] = [f_r, f_s \circ M_2 M_1^t], \tag{5.14}$$

hence it suffices to show that for all a, b, c, d,

$$\Psi(a,b,c,d;r,s) := [f_r(x,y), f_s(ax+by, cx+dy)] \ge 0 \tag{5.15}$$

A calculation shows that

$$\Psi(a,b,c,d;r,s) = a^4 + b^4 + c^4 + d^4 + \\ 6r(a^2b^2 + c^2d^2) + 6s(a^2c^2 + b^2d^2) + 6rs(a^2d^2 + 4abcd + b^2c^2). \tag{5.16}$$

When $s = U(r)$, a sos expression can be found:

$$2(1-r)\Psi(a,b,c,d;r,U(r)) = (1+r)(1+3r)(a^2+b^2-c^2-d^2)^2 \\ -4r(a^2+c^2-b^2-d^2)^2 + (1+r)(1-3r)(a^2+d^2-b^2-c^2)^2 \\ -8r(1+3r)(ab+cd)^2, \tag{5.17}$$

which is non-negative when $r \in [-\frac{1}{3}, 0]$. Note that $\Psi(1,1,1,-1; r, U(r)) = 0$; reaffirming that $[f_r, g_{U(r)}] = 0$. □

Theorem 5.5. *Suppose $r, s \in [-\frac{1}{3}, 0]$. If $s \ge U(r)$, $p \in [[f_r]]$ and $q \in [[f_s]]$, then $[p, q] \ge 0$. If $s < U(r)$, then there exist $p \in [[f_r]]$ and $q \in [[f_s]]$ so that $[p, q] < 0$.*

Proof. If $0 \ge s \ge U(r)$, then $s \in [U(r), T(U(r))]$, hence f_s is a convex combination of $f_{U(r)}$ and $f_{T(U(r))}$, and each $f_s \circ M$ is a convex combination of $f_{U(r)} \circ M$ and $f_{T(U(r))} \circ M$. By Theorem 5.4, $[f_r, f_s \circ M]$ is a convex combination of non-negative numbers and is non-negative. If $U(r) > s \ge -\frac{1}{3}$, then $[f_r, g_s] < 0$ by (5.6). □

We now have the tools to analyze $B \in \mathcal{B}_{2,4}$. If $Q_{2,4} \subseteq B \subseteq P_{2,4}$, let

$$\Delta(B) = \{\lambda \in \mathbb{R} : f_\lambda \in B\}. \tag{5.18}$$

Theorem 5.6. *If $B \subset F_{2,4}$ is a blender, then $\Delta(B) = [\tau, T(\tau)]$ for some $\tau \in [-\frac{1}{3}, 0]$.*

Proof. By (P2), $\Delta(B)$ is a closed interval. We have seen that $\Delta(P_{2,4}) = [-\frac{1}{3}, \infty)$. Since $Q_{2,4} = P^*_{2,4} = \Sigma^*_{2,4}$, by (3.11), $f_\lambda \in Q_{2,4}$ if and only if $\begin{pmatrix} 1 & 0 & \lambda \\ 0 & \lambda & 0 \\ \lambda & 0 & 1 \end{pmatrix}$ is psd; that is, $\Delta(Q_{2,4}) = [0, 1]$. Otherwise, let $\tau = \inf\{\lambda : f_\lambda \in B\}$. Since $Q_{2,4} \subsetneq B \subsetneq P_{2,4}$, $\tau \in (-\frac{1}{3}, 0)$. By (P2), $f_\tau \in B$ and by (P3), $f_{T(\tau)} \in B$, and by convexity, $f_\nu \in B$ for $\nu \in [\tau, T(\tau)]$. If $\nu < \tau$, then $f_\nu \notin B$ by definition. If $\nu > T(\tau)$ and $f_\nu \in B$, then $f_{T(\nu)} \in B$ and $T(\nu) < T(T(\tau)) = \tau$, a contradiction. □

If M is singular, then $f_\lambda \circ M$ is a 4th power; accordingly, for $\tau \in [-\frac{1}{3}, 0]$, let

$$B_\tau := \bigcup_{\tau \le \lambda \le \frac{1}{3}} [[f_\lambda]] = \{p : p \sim f_\lambda, \tau \le \lambda \le \tfrac{1}{3}\} \cup \{(\alpha x + \beta y)^4 : \alpha, \beta \in \mathbb{R}\}. \tag{5.19}$$

Theorem 5.7. *If $B \in \mathcal{B}_{2,4}$, then $B = B_\tau$ for some $\tau \in [-\frac{1}{3}, 0]$ and $B^*_\tau = B_{U(\tau)}$.*

Proof. Suppose B is a blender and $Q_{2,4} \subsetneq B \subsetneq P_{2,4}$. Then $\Delta(B) = [\tau, T(\tau)]$ by Theorem 5.6, so $B = B_\tau$ by Prop. 5.2. We need to show that each such B_τ is a blender. Since $B_0 = Q_{2,4}$ and $B_{-\frac{1}{3}} = P_{2,4}$ are blenders, we may assume $\tau > -\frac{1}{3}$ and all $p \in B_\tau$ which are not 4th powers are pd. Clearly, (P3) holds in B_τ.

Suppose $p_m \in B_\tau$ and $p_m \to p$. If p is a 4th power, then $p \in B_\tau$. If p is pd, then $K(p_m) \to K(p)$ by (5.11), (5.12) and continuity. In any case, $K(p_m) \ge \phi(\tau)$, so $K(p) \ge \phi(\tau)$ and $p \in B_\tau$. Finally, if $p \in Z_{2,4}$, then $K(p_m) \ge \phi(\tau) > \phi(-\frac{1}{3}) = K(p)$ by Lemma 5.3, and this contradiction completes the proof of (P2).

We turn to (P1). Suppose $p, q \in B_\tau$ and $p + q \notin B_\tau$. Since $p + q$ is pd, $p + q \sim f_\lambda$ for some $\lambda < \tau$, and so there exists M so that $p \circ M + q \circ M = f_\tau$. But now, (5.5) and Theorem 5.5 give a contradiction:

$$0 > [f_\lambda, g_{U(\tau)}] = [p \circ M, g_{U(\tau)}] + [q \circ M, g_{U(\tau)}] \ge 0. \tag{5.20}$$

Thus, $p + q \in B_\tau$ and (P1) is satisfied, showing that B_τ is a blender. It follows from Prop. 2.7 and Theorem 5.5 that $B^*_\tau = B_\nu$ for some ν. But by Theorem 5.5, $B_{U(\tau)} \subseteq B^*_\tau$ and if $\lambda < U(\tau)$, then $f_\lambda \notin B^*_\nu$, thus $B^*_\tau = B_{U(\tau)}$. □

A computation shows that $\phi^2(\lambda)+\phi^2(U(\lambda)) = \frac{1}{27}$, and this gives an alternate way of describing the dual cones. This result was garbled in [19, p.141] into the statement that $B_\tau^* = B_\nu$, where $\tau^2 + \nu^2 = \frac{1}{9}$. The self-dual blender $B_{\nu_0} = B_{\nu_0}^*$ occurs for $\nu_0 = 1 - \sqrt{4/3}$. We know of no other interesting properties of B_{ν_0}.

6. $K_{2,2r}$: binary convex forms

The author's Ph.D. thesis, submitted in 1976 and published as [16, 17] in 1978 and 1979, discussed $N_{n,2r}$. (The identification of $N_{n,2r}$ with $K_{n,2r}$ was not made there.) Unbeknownst to him, V. I. Dmitriev had earlier worked on similar questions at Kharkov University. In 1969, S. Krein, Dmitriev's advisor, asked about the extreme elements of $K_{2,2r}$. Dmitriev wrote [9] in 1973 and [10] in 1991. Dmitriev writes in [10]: "I am not aware of any articles on this topic, except [9]." We have seen both [9] and [10] in Russian and [10] in its English translation, thanks to the diligence of the Interlibrary Loan Staff of the University of Illinois Library. To complicate matters, there are at least two mathematicians named V. I. Dmitriev in MathSciNet; the author of [9, 10] is affiliated with Kursk State Technical University.

Let

$$q_\lambda(x, y) = x^6 + 6\lambda x^5 y + 15\lambda^2 x^4 y^2 + 20\lambda^3 x^3 y^3 + 15\lambda^2 x^2 y^4 + 6\lambda x y^5 + y^6. \quad (6.1)$$

In the language of this paper, the four relevant results from [9, 17, 10] are these:

Proposition 6.1.

(i) $K_{2,4} = Q_{2,4}$.
(ii) $Q_{2,2r} \subsetneq K_{2,2r}$ *for* $r \geq 3$.
(iii) The elements of $\mathcal{E}(K_{2,6})$*, are* $[[q_\lambda]]$*, where* $0 < |\lambda| \leq \frac{1}{2}$.
(iv) $K_{3,4} \subsetneq Q_{3,4}$*; specifically,* $x^4 + y^4 + z^4 + 6x^2y^2 + 6x^2z^2 + 2y^2z^2 \in K_{3,4} \setminus Q_{3,4}$.

Dmitriev [9] gave a proof of (i) and (ii) for even r (using $(x^4 + y^4)^{r/2}$ as the counterexample); his [10] gave a proof of (iii). Prop. 6.1 appeared in [17], but (iii) was announced without proof. (The results from [17] were in the author's thesis.) Note that (i) and (ii) follow from Prop. 3.7 and Theorems 3.9 and 3.10. Since $P_{n,m} = \Sigma_{n,m}$ if $n = 2$ or $(n, m) = (3, 4)$, these examples are not helpful in resolving Parrilo's question about convex forms which are not sos.

The rest of this section discusses $\partial(K_{2,2r})$, mostly for small r. For

$$p(x, y) = \sum_{i=0}^{2r} \binom{2r}{i} a_i x^{2r-i} y^i, \quad (6.2)$$

we define essentially the determinant of the Hessian of p at (x,y). Let

$$\Theta_p(x,y) := \sum_{m=0}^{4r-4} b_m x^{4r-4-m} y^m, \quad \text{where}$$
$$b_m := \sum_{j=0}^{2r-1} \left(\binom{2r-2}{j}\binom{2r-2}{m-j} - \binom{2r-2}{j-1}\binom{2r-2}{m-j+1} \right) a_j a_{m+2-j}, \tag{6.3}$$

with the convention that $a_i = 0$ if $i < 0$ or $i > 2r$.

Proposition 6.2. [10, Prop.B] *Suppose $p \in P_{2,2r}$. Then $p \in K_{2,2r}$ if and only if $\Theta_p \in P_{2,4r-4}$ and $p \in \partial(K_{2,2r})$ if and only if Θ_p is psd but not pd.*

Proof. A direct computation shows that

$$\frac{\partial^2 p}{\partial x^2}\frac{\partial^2 p}{\partial y^2} - \left(\frac{\partial^2 p}{\partial x \partial y}\right)^2 = (2r)^2(2r-1)^2\Theta_p(x,y). \tag{6.4}$$

Since $Hes(p;u,u) = 2r(2r-1)p(u) \ge 0$, the first assertion is proved. Further, $p \in \partial(K_{2,2r})$ if and only if $Hes(p;u_0,v_0) = 0$ for some $u_0 \ne 0, v_0 \ne 0$. □

Observe that $\Theta_{(\alpha\cdot)^{2r}} = 0$, and if $q(x,y) = p(ax+by, cx+dy)$, then it may be checked $\Theta_q(x,y) = (ad-bc)^2\Theta_p(ax+by,cx+dy)$. Thus, if $q \in \partial(K_{2,2r})$, we may assume that $q \sim p$, where $\Theta_p(0,1) = 0$, so that

$$0 = b_0 = a_0a_2 - a_1^2; \qquad 0 = b_1 = (2r-2)(a_0a_3 - a_1a_2). \tag{6.5}$$

We prove that $K_{2,4} = Q_{2,4}$, using the argument of [17] and, essentially, [9].

Proposition 6.3. $K_{2,4} = Q_{2,4}$.

Proof. Suppose $q \in \mathcal{E}(K_{2,4})$. Then $q \in \partial(K_{2,4})$ and $q \sim p$ where Θ_p is psd, but $\Theta_p(0,1) = 0$. If $a_0 = 0$, then $p(0,1) = 0$, so by Prop. 4.1, $p(x,y) = a_4y^4$ is a 4th power. Otherwise, $a_0 > 0$, and if we write $a_1 = ra_0$, then by (6.5), we have $a_2 = r^2a_0$ and $a_3 = r^3a_0$. Write $a_4 = r^4a_0 + s$. A computation shows that $\Theta_p(x,y) = a_0sx^2(x+ry)^2$, hence $s \ge 0$ and $p(x,y) = a_0(x+ry)^4 + sy^4$. Since $Q_{2,4} \subset K_{2,4}$ and $s \ge 0$, it follows that $p \in \mathcal{E}(K_{2,4})$ if and only if $s = 0$. Thus $p \in K_{2,4}$, being a sum of extremal elements, is a sum of 4th powers. □

If $2r = 6$, then we shall need $\Theta_p(x,y)$ in full bloom:

$$\begin{aligned}\Theta_p(x,y) = (a_0a_2 - a_1^2)x^8 + 4(a_0a_3 - a_1a_2)x^7y + (6a_0a_4 + 4a_1a_3 - 10a_2^2)x^6y^2 \\ +4(a_0a_5 + 4a_1a_4 - 5a_2a_3)x^5y^3 + (a_0a_6 + 14a_1a_5 + 5a_2a_4 - 20a_3^2)x^4y^4 \\ +4(a_1a_6 + 4a_2a_5 - 5a_3a_4)x^3y^5 + (6a_2a_6 + 4a_3a_5 - 10a_4^2)x^2y^6 \\ +4(a_3a_6 - a_4a_5)xy^7 + (a_4a_6 - a_5^2)y^8.\end{aligned} \tag{6.6}$$

Lemma 6.4. *If $p \in K_{2,6}$ and $\Theta_p(x,y) = \ell^2(x,y)B_p(x,y)$, where ℓ is linear and B_p is a pd sextic, then $p \notin \mathcal{E}(K_{2,6})$.*

Proof. After a linear change, we may assume $\ell(x, y) = y$, and assume p is given by (6.2), so that (6.6) holds. Our goal is to show that B_p being pd implies that $p \pm \epsilon y^6$ is convex for small ϵ, which contradicts p being extremal. If $a_0 = p(1, 0) = 0$, then as in Prop. 6.3, $p(x, y) = a_6 y^6$ and $\Theta_p(x, y) = 0$. Otherwise, we again have $a_1 = ra_0$, $a_2 = r^2 a_0$ and $a_3 = r^3 a_0$. A computation shows that

$$\begin{aligned} B_p(x,y) = 6a_0(a_4 - r^4 a_0)x^6 + 4a_0(a_5 + 4ra_4 - 5r^5 a_0)x^5 y \\ +a_0(a_6 + 14ra_5 + 5r^2 a_4 - 20r^6 a_0)x^4 y^2 \\ +4ra_0(a_6 + 4ra_5 - 5r^2 a_4)x^3 y^3 + (6r^2 a_0 a_4 + 4r^3 a_0 a_5 - 10a_4^2)x^2 y^4 \\ +4(r^3 a_0 a_6 - a_4 a_5)xy^5 + (a_4 a_6 - a_5^2)y^6. \end{aligned} \tag{6.7}$$

Observe that if $p_\lambda = p + \lambda y^6$, then a_6 is replaced above by $a_6 + \lambda$ and

$$B_{p_\lambda} = B_p + \lambda(a_0 x^4 y^2 + 4ra_0 x^3 y^3 + 6r^2 a_0 x^2 y^4 + 4r^3 a_0 xy^5 + a_4 y^6). \tag{6.8}$$

Since B_p is pd, there exists sufficiently small ϵ so that $B_{p_{\pm\epsilon}}$ is psd, so $p_{\pm\epsilon} \in K_{2,6}$. But then $p = \frac{1}{2}(p_\epsilon + p_{-\epsilon})$ is not extremal. □

Proof of Prop. 6.1(iii). By Prop. 6.2 and Lemma 6.4, we may assume that $\Theta_p = y^2 B_p$ and B_p is psd, but not pd. If $B_p(0, 1) = 0$, then by (6.7), $a_4 = r^4 a_0$ and $a_5 = r^5 a_0$ and, as before, if $a_6 = r^6 a_0 + t$, then $\Theta_p = atx^4(x + ry)^4$, so $t \ge 0$ and $p \in \mathcal{E}(K_{2,6})$ if and only if $t = 0$, so p is a 6th power.

If $B_p(1, e) = 0$ and $e \ne 0$, and $\tilde{p}(x, y) = p(y, x + ey)$, then $\Theta_{\tilde{p}}(x, y) = 0$ at $(x, y) = (1, 0), (0, 1)$, and by dropping the tilde, we may assume from (6.6) that $0 = a_4 a_6 - a_5^2 = a_3 a_6 - a_4 a_5$. Again, $a_6 = p(0, 1) \ge 0$, and if $a_6 = 0$, then p is a 6th power. Otherwise, we set $a_5 = sa_6$, so that $a_4 = s^2 a_6$ and $a_3 = s^3 a_6$; recall that $a_3 = r^3 a_0$ as well. If $s = 0$, then $a_3 = 0$, so $r = 0$ and $p(x, y) = a_0 x^6 + a_6 y^6$, which is only extremal if it is a 6th power. Thus $s \ne 0$, and similarly, $r \ne 0$. Letting $t = s^{-1}$ and $a_0 = 1$, we obtain the formulation of [10]:

$$p(x,y) = x^6 + 6rx^5 y + 15r^2 x^4 y^2 + 20r^3 x^3 y^3 + 15r^3 tx^2 y^4 + 6r^3 t^2 xy^5 + r^3 t^3 y^6 \tag{6.9}$$

Send $(x, y) \mapsto (a_0^{-1/6} x, a_0^{-1/6}(rt)^{-1/2} y)$ and set $\lambda = \sqrt{r/t} = \sqrt{rs}$ to obtain q_λ.

We still need to show that q_λ is convex! A calculation shows that

$$\Theta_{q_\lambda}(x, y) = (1 - \lambda^2)x^2 y^2 C_\lambda(x, y), \tag{6.10}$$

where

$$\begin{aligned} C_\lambda(x,y) \\ = 6\lambda^2(x^4 + y^4) + (4\lambda + 20\lambda^3)(x^3 y + xy^3) + (1 + 15\lambda^2 + 20\lambda^4)x^2 y^2. \end{aligned} \tag{6.11}$$

Note that

$$\begin{aligned} D_\lambda(x,y) := C_\lambda(x + y, x - y) = (1 + \lambda)(1 + 2\lambda)(1 + 5\lambda + 10\lambda^2)x^4 \\ -2(1 - \lambda^2)(1 - 20\lambda^2)x^2 y^2 + (1 - \lambda)(1 - 2\lambda)(1 - 5\lambda + 10\lambda^2)x^4. \end{aligned} \tag{6.12}$$

If Θ_{q_λ} is psd, then $6\lambda^2(1 - \lambda^2) \ge 0$, so $|\lambda| \le 1$. Under this assumption, it suffices to determine when D_λ is psd. Since $D_\lambda(1, 0), D_\lambda(0, 1) \ge 0$, $|\lambda| \le \frac{1}{2}$. If $D_\lambda(x, y) = E_\lambda(x^2, y^2)$, then the discriminant of the quadratic E_λ is $128\lambda^2(1 - \lambda^2)(1 - 10\lambda^2)$, hence D_λ is psd if $0 \le \lambda^2 \le \frac{1}{10}$. But, if $\frac{1}{20} \le \lambda^2 \le \frac{1}{4}$, then D_λ is a sum of psd terms. Thus D_λ is psd if $|\lambda| \le \frac{1}{2}$; this is also true for C_λ and Θ_{q_λ}, so $q_\lambda \in K_{2,6}$. $\square$

Note that Θ_{q_λ} has two double zeros when $|\lambda| < \frac{1}{2}$, but $\Theta_{q_{1/2}}$ has three double zeros; it is $\frac{9}{8}x^2y^2(x+y)^2(x^2+xy+y^2)$. It seems likely that for $r \ge 3$, the structure of Θ_p for $p \in \mathcal{E}(K_{2,2r})$ will be complicated and $\mathcal{E}(K_{2,2r})$ will be hard to analyze.

Note also that

$$\begin{aligned} q_\lambda(x+y, x-y) = 2(1+\lambda)(1+5\lambda+10\lambda^2)x^6 + 30(1-\lambda^2)(1+2\lambda)x^4y^2 \\ +30(1-\lambda^2)(1-2\lambda)x^2y^4 + 2(1-\lambda)(1-5\lambda+10\lambda^2)y^6. \end{aligned} \tag{6.13}$$

One of the two boundary examples is $q_{-1/2}(x+y, x-y) = x^6 + 45x^2y^4 + 18y^6$, which scales to $x^6 + 15\alpha x^2y^4 + y^6$, where $\alpha^3 = \frac{1}{12}$.

In an attempt to visualize these blenders, we now consider the sections of $P_{2,6} = \Sigma_{2,6}$, $Q_{2,6}$ and $K_{2,6}$ consisting of the normalized even sextic forms

$$g_{A,B}(x,y) = x^6 + \binom{6}{2}Ax^4y^2 + \binom{6}{4}Bx^2y^4 + y^6, \tag{6.14}$$

and identify $g_{A,B}$ with the point (A, B) in the plane.

If $g_{A,B}$ is on the boundary of the $P_{2,6}$ section, then it is psd but not pd, and we may assume $(x+ry)^2 \mid g_{A,B}$ for some $r \ne 0$. Thus, $(x-ry)^2 \mid g_{A,B}$ as well, and since the remaining factor must be even, the coefficients of x^6, y^6 force it to be $x^2 + \frac{1}{r^4}y^2$. Thus, the boundary forms for the section of $P_{2,6}$ are

$$(x^2 - r^2y^2)^2(x^2 + \tfrac{1}{r^4}y^2) = x^6 + (\tfrac{1}{r^4} - 2r^2)x^4y^2 + (r^4 - \tfrac{2}{r^2})x^2y^4 + y^6. \tag{6.15}$$

The parameterized boundary curve

$$(A, B) = \tfrac{1}{15}(\tfrac{1}{r^4} - 2r^2, r^4 - \tfrac{2}{r^2}) \tag{6.16}$$

is strictly decreasing as we move from left to right, and is a component of the curve $500(A^3 + B^3) = 1875(AB)^2 + 150AB - 1$.

By (3.11), $g_{A,B}$ is in $Q_{2,6} = \Sigma^*_{2,6}$, if and only if $\begin{pmatrix} 1 & 0 & A & 0 \\ 0 & A & 0 & B \\ A & 0 & B & 0 \\ 0 & B & 0 & 1 \end{pmatrix}$ is psd if and only if $A \ge B^2$ and $B \ge A^2$, so the section is the region between these two parabolas.

Except for the fortuitous identity (6.13), it would have been very challenging to determine the section for $K_{2,6}$. Scale x and y in (6.13) to get $g_{A,B}$: the parameterization of the boundary is $(\psi(\lambda), \psi(-\lambda))$, where

$$\psi(\lambda) = \frac{(1-\lambda)^{2/3}(1+\lambda)^{1/3}(1+2\lambda)}{(1+5\lambda+10\lambda^2)^{2/3}(1-5\lambda+10\lambda^2)^{1/3}}. \tag{6.17}$$

The intercepts occur when $\lambda = \pm\frac{1}{2}$ and are $(12^{-\frac{1}{3}}, 0)$ and $(0, 12^{-\frac{1}{3}})$. The point $(1,1)$ ($\lambda = 0$) is smooth but has infinite curvature. The Taylor series of $\psi(\lambda)$ at $\lambda = 0$ begins $1 + \frac{16}{3}\lambda^3 - 48\lambda^4$, so $x - y \approx \frac{32}{3}\lambda^3$ and $x + y - 2 \approx -96\lambda^4$, hence

$$x + y - 2 \approx -\tfrac{3^{7/3}}{2^{5/3}}(x-y)^{4/3}.$$

The maximum value of $\psi(\lambda)$ is $5^{-5/3}(1565 + 496\sqrt{10})^{1/3} \approx 1.000905$ at $\lambda = \frac{2\sqrt{10}-5}{15} \approx .0883$; this was asserted without proof in [17, p.232].

We conclude with a description of the trinomials in $\partial(K_{2,2r})$. Suppose $1 \le v \le 2r-1$, $a, c > 0$ and suppose

$$h(x,y) = ax^{2r} + bx^{2r-v}y^v + cy^{2r} \in K_{2,2r}. \tag{6.18}$$

An examination of the end terms of Θ_h shows that v must be even and $b \ge 0$. If $b = 0$, then $h \in Q_{2,2r}$, so we assume $b > 0$, and wish to find the largest possible value of b. Calculations, which we omit, show that if

$$\begin{gathered} h_{r,k}(x,y) := (r-k)(2(r-k)-1)^2 x^{2r} \\ +r(2r-1)(2k-1)(2r-2k-1)x^{2r-2k}y^{2k} + k(2k-1)^2 y^{2r}, \end{gathered} \tag{6.19}$$

then $\Theta_{h_{r,k}}(x,y) = x^{2r-2-2k}y^{2k-2}(x^2-y^2)^2 g(x,y)$, where g is a (psd) sum of even terms, and that if $c > 0$ and $g_{r,k,c} = h_{r,k} + cx^{2r-2k}y^{2k}$, then $\Theta_{g_{r,k,c}}(1,1) < 0$. Given (a,c), there exist (α, β) so that the coefficients of x^{2r} and y^{2r} in $h_{r,k}(\alpha x, \beta y)$ are both 1, and we obtain the examples given in [17, Prop.1]. In particular,

$$h_{4k,2k}(x,y) \sim x^{4k} + (8k-2)x^{2k}y^{2k} + y^{4k} \in \partial(K_{2,4k}). \tag{6.20}$$

Similar methods show that

$$x^{6k} + (6k-1)(6k-3)x^{4k}y^{2k} + (6k-1)(6k-3)x^{2k}y^{4k} + y^{6k} \in \partial(K_{2,6k}). \tag{6.21}$$

We have been unable to analyze $K_{2,8}$ completely, but have found this interesting element in $\mathcal{E}(K_{2,8})$:

$$p(x,y) = (x^2+y^2)^4 + \tfrac{8}{\sqrt{7}}\, xy(x^2-y^2)(x^2+y^2)^2, \tag{6.22}$$

for which $\Theta_p(x,y) = 3072x^2(x-y)^2y^2(x+y)^2(x^2+y^2)^2$.

7. Sums of 4th powers and binary octic forms

Hilbert's 17th Problem asks whether $p \in P_{n,2r}$ must be a sum of squares of rational functions: does there exist $h = h_p \in F_{n,d}$ (for some d) so that $h^2 p \in \Sigma_{n,2r+2d} = W_{n,2(r+d)}$? Artin proved that the answer is "yes". (See [21, 23].) Becker [2] investigated the question for higher even powers. His result implies that if $p \in P_{2,2kr}$ and all real linear factors of p (if any) occur to an exponent which is a multiple of $2k$, then there exists $h = h_p \in F_{2,d}$ (for some d) so that $h^{2k}p \in W_{2,(r+d,2k)}$.

By Becker's criteria, f_λ (c.f. (5.2)) is a sum of 4th powers of rational functions if and only if it is pd; that is, $\lambda \in (-\frac{1}{3}, \infty)$. As we have seen, $f_\lambda \in Q_{2,4} = W_{2,(1,4)}$ if and only if $\lambda \in [0,1]$. If ℓ is linear and $\ell^4 f = \sum_k h_k^4 \in W_{2,(2,4)}$, then $\ell | h_k$, so if $f_\lambda \notin Q_{2,4}$ and $h^4 f \in W_{2,(1+d,4)}$, then $\deg h = d \ge 2$. The identity

$$\begin{gathered} 3(3x^4 - 4x^2y^2 + 3y^4)(x^2+y^2)^4 \\ = 2((x-y)^4 + (x+y)^4)(x^8+y^8) + 5x^{12} + 11x^8y^4 + 11x^4y^8 + 5y^{12} \end{gathered} \tag{7.1}$$

shows that $(x^2+y^2)^4 f_\lambda \in W_{2,(3,4)}$ for $\lambda \in [-\frac{2}{9}, \frac{11}{3}]$, since $T(-\frac{2}{9}) = \frac{11}{3}$, c.f. (5.4).

We offer the following conjectural characterization of $W_{2,(u,4)}$:

Conjecture 7.1. *If $p \in P_{2,4u}$, then $p \in W_{2,(u,4)}$ if and only if there exist $f, g \in P_{2,2u}$ so that $p = f^2 + g^2$.*

It follows from (1.18) that the square of a psd binary form is a sum of three 4th powers. Conjecture 7.1 thus implies that any sum of 4th powers of polynomials is a sum of six 4th powers of polynomials. If $p \in W_{2,(u,4)}$, then $p \in P_{2,4u} = \Sigma_{2,4u}$, so $p = f^2 + g^2$ for some $f, g \in F_{n,2u}$; the conjecture says that there is a representation in which f and g are themselves psd.

This seems related to a result in [6] about sums of 4th powers of rational functions over real closed fields. If $p = \sum h_k^4$ and $\ell | p$ for a linear form, then $\ell^{4t} | p$ for some t and $\ell^t | h_k$, so we may assume p is pd. The following is a special case of [6, Thm.4.12], referring to sums of 4th powers of non-homogeneous rational functions.

Proposition 7.2. *Suppose $p \in \mathbb{R}[x]$ is pd. Then p is a sum of 4th powers in $\mathbb{R}(x)$ if and only if there exist pd f, g, h in $\mathbb{R}[x]$, $\deg f = \deg g$, such that $h^2 p = f^2 + g^2$.*

It follows that a sum of 4th powers in $\mathbb{R}(x)$ is a sum of at most six 4th powers.

Theorem 7.3. *Conjecture 7.1 is true for $p \in W_{2,(1,4)} = Q_{2,4}$.*

Proof. We have seen that if $p \in W_{2,(1,4)}$, then $p \sim f_\lambda$ for $\lambda \in [0,1]$. If $\lambda \in (\frac{1}{3}, 1]$, then $T(\lambda) \in [0, \frac{1}{3})$, so it suffices to find a representation for f_λ with $\lambda \in [0, \frac{1}{3}]$. Such a representation is $f_\lambda(x,y) = (x^2 + 3\lambda y^2)^2 + (1 - 9\lambda^2)(y^2)^2$. □

Theorem 7.4. *Conjecture 7.1 is true for even symmetric octics.*

It will take some work to get to the proof of Theorem 7.4. For the rest of this section, write $W := W_{2,(2,4)}$. We first characterize $\partial(W^*)$.

Theorem 7.5. *If $p \in \partial(W^*)$, then $p = (\alpha\cdot)^8$ or $p \sim q$, where*

$$q(x,y) = d_0x^8 + 8d_1x^7y + 28d_2x^6y^2 + 28d_6x^2y^6 + 8d_7xy^7 + d_8y^8, \tag{7.2}$$

and the following form is psd:

$$(6d_2u^2 + 6d_6w^2)(d_0u^4 + 4d_2u^3w + 4d_6uw^3 + d_8w^4) - (2d_1u^3 + 2d_7w^3)^2. \tag{7.3}$$

Proof. Consider a typical element $q \in W^*$,

$$q(x,y) = \sum_{k=0}^{8} \binom{8}{k} d_k x^{8-k} y^k. \tag{7.4}$$

Then as in Prop. 3.8,

$$\begin{aligned} H_q(u,v,w) := [q, (ux^2 + vxy + wy^2)^4] = d_0u^4 + 4d_1u^3v + d_2(6u^2v^2 + 4u^3w) \\ + d_3(4uv^3 + 12u^2vw) + d_4(v^4 + 12uv^2w + 6u^2w^2) + d_5(4v^3w + 12uvw^2) \\ + d_6(6v^2w^2 + 4uw^3) + 4d_7vw^3 + d_8w^4 \end{aligned} \tag{7.5}$$

is a psd ternary quartic in u, v, w. If $q \in \partial(W^*)$, then $[q, h^2] = 0$ for some non-zero quadratic h. Since $\pm h \sim x^2, xy, x^2 + y^2$, it suffices by Prop. 2.6 to consider three cases: $[q, x^8] = 0, [q, x^4y^4] = 0$ and $[q, (x^2 + y^2)^4] = 0$. Since

$$420(x^2 + y^2)^4 = 256(x^8 + y^8) + \sum_{\pm}(x \pm \sqrt{3}y)^8 + (\sqrt{3}x \pm y)^8, \tag{7.6}$$

$[q, (x^2+y^2)^4] = 0$ implies that $q(1, 0) = q(0, 1) = q(1, \pm\sqrt{3}) = q(\sqrt{3}, \pm 1) = 0$; since q is psd, $q = 0$. (An alternate proof derives this result from $(x^2 + y^2)^4 \in int(Q_{2,8})$ by [19, Thm.8,15(ii)], so $(x^2 + y^2)^4 \in int(W)$.)

Suppose $[h, (x^2)^4] = 0$; that is, $H_q(1, 0, 0) = 0$. Then $d_0 = 0$, and since H_q is now at most quadratic in u, it follows that $d_1 = d_2 = 0$. This implies that the coefficient of u^2 in H_q is $12d_3vw + 6d_4w^2$, hence $d_3 = 0$ and

$$\begin{aligned} H_q(u, v, w) = u^2(6d_4w^2) + 2u(2d_6w^3 + 6d_5vw^2 + 6d_4v^2w) \\ +(d_8w^4 + 4d_7w^3v + 6d_6w^2v^2 + 4d_5wv^3 + d_4v^4). \end{aligned} \tag{7.7}$$

Since H_q is psd if and only if its discriminant with respect to u is psd in v, w, and this discriminant is $-30d_4^2v^4w^2+$ lower terms in v, $d_4 = 0$. Since H_q cannot be linear in u, it follows that $d_5 = d_6 = 0$ and $H_q(u, v, w) = d_8w^4 + 4d_7w^3v$, which is only psd if $d_7 = 0$, so that $q(x, y) = d_8y^8$ is an 8th power.

Finally, suppose $[q, x^4y^4] = 0$; that is, $H_q(0, 1, 0) = d_4 = 0$. Since H_q is at most quadratic in v, it follows that $d_3 = d_5 = 0$, so q has the shape (7.2) and

$$\begin{aligned} H_q(u, v, w) = v^2(6d_2u^2 + 6d_6w^2) \\ +2v(2d_1u^3 + 2d_7w^3) + d_0u^4 + 4u^3wd_2 + 4uw^3d_6 + d_8w^4; \end{aligned} \tag{7.8}$$

H_q is psd if and only if its discriminant with respect to v, namely (7.3), is psd. □

It should be possible to characterize $\mathcal{E}(W^*)$, though we do not do so here. One family of extremal elements in $\mathcal{E}(W^*)$ is parameterized by $\alpha \in \mathbb{R}$:

$$\omega_\alpha(x, y) := x^8 + 28x^2y^6 + 24\alpha xy^7 + 3(1 + 2\alpha^2)y^8 \in \mathcal{E}(W^*). \tag{7.9}$$

In this case,

$$\begin{aligned} H_{\omega_\alpha}(u, v, w) &= 6v^2w^2 + 12\alpha vw^3 + u^4 + 4uw^3 + (3 + 6\alpha^2)w^4 \\ &= 6(vw + \alpha w^2)^2 + (u + w)^2(u^2 - 2uw + 3w^2) \end{aligned} \tag{7.10}$$

is psd; $H_{\omega_\alpha}(0, 1, 0) = H_{\omega_\alpha}(1, \alpha, -1) = 0$, and $H_{\omega_\alpha}(u, v, 0) = u^4$ has a 4th order zero at $(0, 1, 0)$. It is unclear whether ω_α has other interesting algebraic properties.

We now limit our focus to the section of even symmetric octics. Let

$$\widetilde{F} = \{((A, B, C)) := Ax^8 + Bx^6y^2 + Cx^4y^4 + Bx^2y^6 + Ay^8 \ : \ A, B, C \in \mathbb{R}\}. \tag{7.11}$$

denote the cone of even symmetric octics, and let

$$\widetilde{W} = W \cap \widetilde{F}. \tag{7.12}$$

Then $\widetilde{W}$ is no longer a blender, because (P3) fails spectacularly. However, it is still a closed convex cone. We give the inner product explicitly:

$$p_i = ((A_i, B_i, C_i)) \implies [p_1, p_2] = A_1A_2 + \tfrac{B_1B_2}{28} + \tfrac{C_1C_2}{70} + \tfrac{B_1B_2}{28} + A_1A_2. \quad (7.13)$$

Let $(\widetilde{W})^* \subset \widetilde{F}$ denote the dual cone to $\widetilde{W}$. Here is a special case of [19, p.142].

Theorem 7.6. *$(\widetilde{W})^* = W^* \cap \widetilde{F}$.*

Proof. Suppose $p \in \widetilde{W}$ and $q \in W^* \cap \widetilde{F}$. Then $p \in W$ and $q \in W^*$ imply $[p, q] \ge 0$, so $q \in (\widetilde{W})^*$. Suppose now that $q \in (\widetilde{W})^*$; we wish to show that $q \in W^*$. Choose $r \in W$, and let $r_1 = r$, $r_2(x, y) = r(x, -y)$, $r_3(x, y) = r(y, x)$ and $r_4(x, y) = r(y, -x)$. Since $q \in \widetilde{F}$, $[r_j, q] = [r, q]$ for $1 \le j \le 4$; since $p = r_1 + r_2 + r_3 + r_4 \in \widetilde{W}$, $0 \le [p, q] = 4[r, q]$. Thus, $[r, q] \ge 0$ as desired. □

We need not completely analyze $(\widetilde{W})^*$ to determine $\widetilde{W}$. The following suffices.

Lemma 7.7. *If $q = ((1, 0, 0))$, $((4, 28, 0))$ or $((6 - 4\lambda^2 + 3\lambda^4, 28(6 - \lambda^2), 420))$, $\lambda \in \mathbb{R}$, then $q \in W^*$.*

Proof. Using the notation of (7.4), suppose

$$q(x, y) = ((d_0, 28d_2, 70d_4)) = d_0^8 + 28d_2x^6y^2 + 70d_4x^4y^4 + 28d_2x^2y^6 + d_0y^8. \quad (7.14)$$

Comparison with (7.13) shows that

$$q \in \widetilde{W}^* \iff ((A, B, C)) \in \widetilde{W} \implies 2d_0A + 2d_2B + d_4C \ge 0. \quad (7.15)$$

On the other hand, (7.5) and Theorem 7.6 imply that $q \in \widetilde{W}^*$ if and only if

$$\begin{aligned} &H_q(u, v, w) \\ &= d_0(u^4 + w^4) + d_2(u^2 + w^2)(6v^2 + 4uw) + d_4(v^4 + 12uv^2w + 6u^2w^2) \end{aligned} \quad (7.16)$$

is psd. If $(d_0, d_2, d_4) = (1, 0, 0)$, then $H_q(u, v, w) = u^4 + w^4$, which is psd, and if $(d_0, d_2, d_4) = (4, 1, 0)$, then

$$H_q(u, v, w) = 4(u + w)^2(u^2 - uw + w^2) + 6(u^2 + w^2)v^2. \quad (7.17)$$

Finally, if $(d_0, d_2, d_4) = (6 - 4\lambda^2 + 3\lambda^4, 6 - \lambda^2, 6)$, then a computation gives

$$\begin{aligned} 2H_q(u, v, w) &= 2(6 - 4\lambda^2 + 3\lambda^4)(u^4 + w^4) \\ &+2(6 - \lambda^2)(u^2 + w^2)(6v^2 + 4uw) + 12(v^4 + 12uv^2w + 6u^2w^2) \\ &= 48(u + w)^2v^2 + 4\lambda^2(u + w)^4 + 3\lambda^4(u^2 - w^2)^2 \\ &+3(2v^2 + 2(u + w)^2 - \lambda^2(u^2 + w^2))^2. \end{aligned} \quad (7.18)$$

Note that $H_q(1, \pm\lambda, -1) = 0$. □

An important family of elements in $\widetilde{W}$ is

$$\begin{aligned} \psi_\lambda(x, y) &:= \tfrac{1}{2}\left((x^2 + \lambda xy - y^2)^4 + (x^2 - \lambda xy - y^2)^4\right) \\ &= ((1,\ 6\lambda^2 - 4,\ \lambda^4 - 12\lambda^2 + 6)). \end{aligned} \quad (7.19)$$

Theorem 7.8. *The extremal elements of* $\widetilde{W}$ *are* x^4y^4 *and* $\{\psi_\lambda : \lambda \ge 0\}$. *Hence* $p = ((A,B,C)) \in \widetilde{W}$ *if and only if*

$$A = B = 0,\ C \ge 0,\ \text{or } A > 0,\ B \ge -4A,\ 36AC \ge B^2 - 64AB - 56A^2. \tag{7.20}$$

Proof. By Lemma 7.7 and (7.15), if $p \in \widetilde{W}$, then $A \ge 0$, $A + 4B \ge 0$ and

$$2(6 - 4\lambda^2 + 3\lambda^4)A + 2(6 - \lambda^2)B + 6C \ge 0. \tag{7.21}$$

We have $A = p(1,0) = p(0,1) \ge 0$, and if $A = 0$ and $p = \sum h_k^4$, then $xy|h_k$, hence $p = [0,0,C]$ with $C \ge 0$. Otherwise, assume that $A = 1$, so that (7.20) becomes

$$B \ge -4, \quad C \ge \tfrac{1}{36}(B^2 - 64B - 56). \tag{7.22}$$

The first inequality follows from $((4,28,0)) \in \widetilde{W}^*$, and we can thus write $B = 6\alpha^2 - 4$, where $\alpha = \sqrt{\frac{B+4}{6}}$. Put $\lambda = \alpha$ in (7.21) to obtain

$$C \ge \alpha^4 - 12\alpha^2 + 6 = \tfrac{1}{36}(B^2 - 64B - 56). \tag{7.23}$$

Suppose $p = ((A,B,C))$ satisfies (7.20). If $A = 0$, then $p = cx^4y^4 \in \widetilde{W}$. If $A > 0$, take $A = 1$ and substitute $B = 6\alpha^2 - 4$, so that, by (7.23),

$$p = ((1,B,C)) = ((1, 6\alpha^2 - 4, \alpha^4 - 12\alpha^2 - 6)) + ((0,0,\gamma)) = \psi_\lambda(x,y) + \gamma x^4y^4 \tag{7.24}$$

for some $\gamma \ge 0$, hence $p \in \widetilde{W}$. □

Taking $(A,B) = (1,0)$, we obtain (1.19). Suppose $\lambda, \mu \ge -2$. Then Theorem 7.6 implies that (c.f. (5.2)) $f_\lambda(x,y)f_\mu(x,y) \in W$ if and only if

$$(17 - 12\sqrt{2})(\lambda + 2) \le \mu + 2 \le (17 + 12\sqrt{2})(\lambda + 2) \tag{7.25}$$

There is a peculiar resonance with the example after Theorem 4.7.

Proof of Theorem 7.4. Suppose $((A,B,C))$ satisfies (7.20). If $A = 0$, then $B = 0$ and $((0,0,C)) = C(x^2y^2)^2$. Otherwise, suppose $A = 1$ and write $B = 6\alpha^2 - 4$, so

$$B = 6\alpha^2 - 4, \quad C = \tfrac{1}{36}(B^2 - 64B - 56) + T = \alpha^4 - 12\alpha^2 + 6 + T, \quad T \ge 0. \tag{7.26}$$

Observe that

$$\begin{gathered}(x^4 + (3\alpha^2 - 2)x^2y^2 + y^4)^2 + (T - 8\alpha^4)(x^2y^2)^2 \\ = ((1, 6\alpha^2 - 4, 9\alpha^4 - 12\alpha^2 + 6)) + ((0,0,T - 8\alpha^4)) = ((1,B,C)),\end{gathered} \tag{7.27}$$

so if $T \ge 8\alpha^4$, then we are done. If $0 \le T \le 8\alpha^4$, note that

$$\begin{gathered}\tfrac{1}{2}\left(\left((x^2 - \sqrt{\lambda}xy - y^2)^2 + \mu x^2y^2\right)^2 + \left((x^2 + \sqrt{\lambda}xy - y^2)^2 + \mu x^2y^2\right)^2\right) \\ = ((1, 6\lambda + 2\mu - 4, 6 - 12\lambda + \lambda^2 - 4\mu + 2\lambda\mu + \mu^2))\end{gathered} \tag{7.28}$$

is a sum of two squares of psd forms if $\mu \ge 0$. One solution to the system

$$\begin{gathered}6\alpha^2 - 4 = 6\lambda + 2\mu - 4, \\ \alpha^4 - 12\alpha^2 + 6 + T = 6 - 12\lambda + \lambda^2 - 4\mu + 2\lambda\mu + \mu^2\end{gathered} \tag{7.29}$$

is

$$\lambda = \frac{3\alpha^2 - \sqrt{\alpha^4 + T}}{2}, \quad \mu = \frac{3(\sqrt{\alpha^4 + T} - \alpha^2)}{2}. \tag{7.30}$$

Evidently, $\mu \geq 0$; since $T \leq 8\alpha^4$, $\lambda \geq 0$, so $\sqrt{\lambda}$ is real. □

8. Bibliography

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Bruce Reznick
Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801
e-mail: `reznick@math.uiuc.edu`