BLENDERS

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Dedicated to the memory of Julius Borcea

Abstract. A blender is a closed convex cone of real homogeneous polynomials that is also closed under linear changes of variable. Non-trivial blenders only occur in even degree. Examples include the cones of psd forms, sos forms, convex forms and sums of $2u$-th powers of forms of degree v. We present some general properties of blenders and analyze the extremal elements of some specific blenders.

1. Introduction and Overview

Let $F_{n,d}$ denote the vector space of real homogeneous forms $p(x_1, \dots, x_n)$ of degree d. A blender is a closed convex cone in $F_{n,d}$ which is also closed under linear changes of variable. Blenders were introduced in [18] to help describe several different familiar cones of polynomials, but that memoir was mainly concerned with the cones of psd and sos forms and their duals, and the discussion of blenders *per se* was scattered there (pp. 36-50, 119-120, 140-142). This paper is devoted to a general discussion of blenders and their properties, as well as considering the extremal elements of some particular blenders not discussed in [18].

Non-trivial blenders will only occur when $d = 2r$ is an even integer. Choi and Lam [3, 4] named the cone of *psd* forms:

$$P_{n,2r} := \{p \in F_{n,2r} : u \in \mathbf{R}^n \implies p(u) \geq 0\}, \tag{1.1}$$

and the cone of *sos* forms:

$$\Sigma_{n,2r} := \left\{ p \in F_{n,2r} : p = \sum_{k=1}^{s} h_k^2,\ h_k \in F_{n,r} \right\}. \tag{1.2}$$

Other blenders of interest in [18] are the cone of sums of $2r$-th powers:

$$Q_{n,2r} := \left\{ p \in F_{n,2r} : p = \sum_{k=1}^{s} (\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2r},\ \alpha_{kj} \in \mathbf{R} \right\} \tag{1.3}$$

and the "Waring blenders": suppose $r = uv$, $u, v \in \mathbf{N}$ and let:

$$W_{n,(u,2v)} := \left\{ p \in F_{n,2r} : p = \sum_{k=1}^{s} h_k^{2v},\ h_k \in F_{n,u} \right\}. \tag{1.4}$$

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Note that $W_{n,(r,2)} = \Sigma_{n,2r}$ and $W_{n,(1,2r)} = Q_{n,2r}$.

The Waring blenders generalize. If $d = 2r$ and $\sum_{i=1}^m u_i v_i = r$, let

$$W_{n,\{(u_1,2v_1),\dots,(u_m,2v_m)\}} := \left\{ p \in F_{n,2r} : p = \sum_{k=1}^{s} h_{k,1}^{2v_1} \cdots h_{k,m}^{2v_m},\ h_{k,i} \in F_{n,u_i} \right\}. \tag{1.5}$$

There has been recent interest in the cones of convex forms:

$$K_{n,2r} := \{p \in F_{n,2r} : p \text{ is convex}\}. \tag{1.6}$$

We shall use the two equivalent definitions of "convex" (see e.g. [23, Thm.4.1,4.5]): under the *line segment* definition, p is convex if for all $u, v \in \mathbf{R}^n$ and $\lambda \in [0,1]$,

$$p(\lambda u + (1-\lambda)v) \le \lambda p(u) + (1-\lambda)p(v). \tag{1.7}$$

The *Hessian* definition says that if

$$Hes(p; u, v) := \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 p}{\partial x_i \partial x_j}(u) v_i v_j, \tag{1.8}$$

then p is convex provided $Hes(p; u, v) \ge 0$ for all $u, v \in \mathbf{R}^n$. The cone $K_{n,m}$ appeared in [18], but as $N_{n,m}$ (see Corollary 4.5). Pablo Parrilo asked whether every convex form is sos; that is, is $K_{n,2r} \subseteq \Sigma_{n,2r}$? This question has been answered by Greg Blekherman [2] in the negative. For fixed n, the "probability" that a convex form is sos goes to 0 as $r \to \infty$. No examples of $p \in K_{n,2r} \setminus \Sigma_{n,2r}$ are yet known.

We now make the definition of blender more precise. Suppose $n \ge 1$ and $d \ge 0$. The index set for monomials in $F_{n,d}$ consists of n-tuples of non-negative integers:

$$\mathcal{I}(n,d) = \left\{ i = (i_1, \dots, i_n) : \sum_{k=1}^{n} i_k = d \right\}. \tag{1.9}$$

Write $N(n,d) = \binom{n+d-1}{n-1} = |\mathcal{I}(n,d)|$ and for $i \in \mathcal{I}(n,d)$, let $c(i) = \frac{d!}{i_1! \cdots i_n!}$ be the associated multinomial coefficient. The abbreviation u^i means $u_1^{i_1} \dots u_n^{i_n}$, where u may be an n-tuple of constants or variables. Every $p \in F_{n,d}$ can be written as

$$p(x_1, \dots, x_n) = \sum_{i \in \mathcal{I}(n,d)} c(i) a(p; i) x^i. \tag{1.10}$$

The identification of p with the $N(n,d)$-tuple $(a(p;i))$ shows that $F_{n,d} \approx \mathbf{R}^{N(n,d)}$ as a vector space. The topology placed on $F_{n,d}$ is the usual one: $p_m \to p$ means that for every $i \in \mathcal{I}(n,d)$, $a(p_m; i) \to a(p; i)$.

For $\alpha \in \mathbf{R}^n$, define $(\alpha\cdot)^d \in F_{n,d}$ by

$$(\alpha\cdot)^d(x) = \left(\sum_{k=1}^{n} \alpha_k x_k \right)^d = \sum_{i \in \mathcal{I}(n,d)} c(i) \alpha^i x^i. \tag{1.11}$$

If α is regarded as a row vector and x as a column vector, then $(\alpha\cdot)^d(x) = (\alpha x)^d$. If $M = [m_{ij}] \in Mat_n(\mathbf{R})$ is a (not necessarily invertible) real $n \times n$ matrix and $p \in F_{n,d}$, we define $p \circ M \in F_{n,d}$ by

$$(1.12) \qquad (p \circ M)(x_1, \dots, x_n) = p(\ell_1, \dots, \ell_n), \qquad \ell_j(x_1, \dots, x_n) = \sum_{k=1}^{n} m_{jk} x_k.$$

If x is viewed as a column vector, then $(p \circ M)(x) = p(Mx)$; $(\alpha\cdot)^d \circ M = (\alpha M \cdot)^d$.

Define $[[p]]$ to be $\{p \circ M : M \in Mat_n(\mathbf{R})\}$, the *closed orbit of* p. If $p = q \circ M$ for *invertible* M, we write $p \sim q$; invertibility implies that $\sim$ is an equivalence relation.

Lemma 1.1.
(i) If $p \in F_{n,d}$ and d is odd, then $p \sim \lambda p$ for every $0 \neq \lambda \in \mathbf{R}$.
(ii) If $p \in F_{n,d}$ and d is even, then $p \sim \lambda p$ for every $0 < \lambda \in \mathbf{R}$.
(iii) If $u, \alpha \in \mathbf{R}^n$, then there exists a (singular) M so that $p \circ M = p(u)(\alpha\cdot)^d$.

Proof. For (i), (ii), observe that $(p \circ (cI_n)) = c^d p$ since p is homogeneous, and cI_n is invertible if $c \neq 0$. For (iii), note that if $m_{jk} = u_j \alpha_k$ for $1 \le j, k \le n$, then

$$(1.13) \quad \ell_j(x) = u_j \sum_{k=1}^{n} \alpha_k x_k = (\alpha x) u_j \implies (p \circ M)(x_1, \dots, x_n) = (\alpha x)^d p(u_1, \dots, u_n)$$

by homogeneity. □

Definition. A set $B \subseteq F_{n,d}$ is a *blender* if these conditions hold:
(P1) If $p, q \in B$, then $p + q \in B$.
(P2) If $p_m \in B$ and $p_m \to p$, then $p \in B$.
(P3) If $p \in B$ and $M \in Mat_n(\mathbf{R})$, then $p \circ M \in B$.

Thus, a blender is a closed convex cone of forms which is also a union of closed orbits. Lemma 1.1 makes it unnecessary to specify in (P1) that $p \in B$ and $\lambda \ge 0$ imply $\lambda p \in B$. Let $\mathcal{B}_{n,d}$ denote the set of blenders in $F_{n,d}$. Trivially, $\{0\}, F_{n,d} \in \mathcal{B}_{n,d}$.

It is simple to see that $P_{n,2r}$ is a blender: conditions (P1) and (P2) can be verified pointwise and if $p(u) \ge 0$ for every u, then the same will be true for $p(Mu)$. Similarly, $K_{n,2r}$ is a blender because (P1) and (P2) follow from the Hessian definition and (P3) follows from the line segment definition.

If $B_1, B_2 \in \mathcal{B}_{n,d}$, then $B_1 \cap B_2 \in \mathcal{B}_{n,d}$. Define the *Minkowski sum*

$$(1.14) \qquad B_1 + B_2 := \{p_1 + p_2 : p_i \in B_i\}.$$

The smallest blender containing both B_1 and B_2 must include $B_1 + B_2$; this set is a blender (Theorem 3.5(i)), but it requires an argument to prove (P2). It is not hard to see that $\mathcal{B}_{n,d}$ is not always a chain. Let $(n, d) = (2, 8)$ and let $B_1 = W_{2,\{(1,6),(1,2)\}}$ and $B_2 = W_{2,\{(1,4),(1,4)\}}$. Then $x^6y^2 \in B_1$ and $x^4y^4 \in B_2$. If $x^6y^2 \in B_2$, then

$$(1.15) \qquad x^6 y^2 = \sum_{k=1}^{s} (\alpha_k x + \beta_k y)^4 (\gamma_k x + \delta_k y)^4.$$

A consideration of the coefficients of x^8 and y^8 shows that $\alpha_k\gamma_k = \beta_k\delta_k = 0$ for all k, hence the only non-zero summands are positive multiples of x^4y^4. Thus $x^6y^2 \notin B_2$, and, similarly, $x^4y^4 \notin B_1$, so $B_1 \setminus B_2$ and $B_2 \setminus B_1$ are both non-empty. It is not clear which octics belong to $B_1 \cap B_2$ and $B_1 + B_2$. If $B_1 \in \mathcal{B}_{n,d_1}$ and $B_2 \in \mathcal{B}_{n,d_2}$, define

$$B_1 * B_2 := \left\{ \sum_{k=1}^{s} p_{1,k}p_{2,k} : p_{i,k} \in B_i \right\}. \tag{1.16}$$

Again, this is a blender (Theorem 3.5(ii)), but (P2) is not trivial to prove.

We review some standard facts about convex cones; see [18, Ch.2,3] and [23]. If $C \subset \mathbf{R}^N$ is a closed convex cone, then $u \in C$ is *extremal* if $u = v_1 + v_2, v_i \in C$, implies that $v_i = \lambda_i u$, $\lambda_i \ge 0$. The set of extremal elements in C is denoted $\mathcal{E}(C)$. All cones $C \ne 0, \mathbf{R}^N$ in this paper have the property that $x, -x \in C$ implies $x = 0$. In such a cone, every element in C is a sum of extremal elements. (It will follow from Prop. 2.4 that if $B \in \mathcal{B}_{n,d}$ and $p, -p \in B$ for some $p \ne 0$, then $B = F_{n,d}$.)

As usual, u is *interior* to C if C contains a non-empty open ball centered at u. The set of interior points of C is denoted $int(C)$, and the boundary of C is denoted $\partial(C)$. The next definition depends on the inner product. If C is a closed convex cone, let

$$C^* = \{v \in \mathbf{R}^N : [u, v] \ge 0 \quad \text{for all} \quad u \in C\}. \tag{1.17}$$

Then $C^* \subset \mathbf{R}^N$ is also a closed convex cone and $(C^*)^* = C$; C and C^* are *dual* cones.

If $u \in C$ (and $\pm x \in C$ implies $x = 0$), then $u \in int(C)$ if and only if $[u, v] > 0$ for every $0 \ne v \in C^*$ (see e.g. [18, p.26]). Thus, if $u \in \partial(C)$ (in particular, if u is extremal), then there exists $v \in C^*$, $v \ne 0$ so that $[u, v] = 0$.

This discussion applies to blenders by identifying $p \in F_{n,d}$ with the $N(n, d)$-tuple of its coefficients. For example, $p \in int(B)$ if there exists $\epsilon > 0$ so that if $|a(q; i)| < \epsilon$ for all $i \in \mathcal{I}(n, d)$, then $p + q \in B$. If $p \sim q \in B$, then p and q simultaneously belong to (or do not belong to) $int(B), \partial(B), \mathcal{E}(B)$. We shall discuss in section two the natural inner product on $F_{n,d}$. It turns out that, under this inner product, $P_{n,2r}$ and $Q_{n,2r}$ are dual cones (Prop. 3.8), as are $K_{n,2r}$ and $W_{n,\{(1,2r-2),(1,2)\}}$ (Theorem 3.11).

The description of $\mathcal{E}(P_{n,2r})$ is extremely difficult if $n \ge 3$. (See e.g [3, 4, 6, 7, 11, 17, 22].) Every element of $\mathcal{E}(\Sigma_{n,2r})$ obviously has the form h^2, but not every square is extremal; e.g.,

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2 = \frac{1}{18}\left((\sqrt{3}\ x + y)^4 + (\sqrt{3}\ x - y)^4 + 16y^4\right). \tag{1.18}$$

We now describe the contents of this paper. Section two reviews the relevant results from [18] regarding the inner product and its many properties. The principal results are that if $B \in \mathcal{B}_{n,d}$ and $B \ne \{0\}, F_{n,d}$, then $d = 2r$ is even and $Q_{n,2r} \subset \pm B \subset P_{n,2r}$ (Prop. 2.5); the dual cone to a blender is also a blender (Prop. 2.7). Section three begins with a number of preparatory lemmas, mainly involving convergence. We show that if B_i are blenders, then so are $B_1 + B_2$ and $B_1 * B_2$ (Theorem 3.5) and hence the Waring blenders and their generalizations are blenders (Theorems 3.6, 3.7). We show that $P_{n,2r}$ and $Q_{n,2r}$ are dual and give a description of $W^*_{n,(u,v)}$ (both from [18]) and show that $K_{n,2r}$ and $W_{n,\{(1,2r-2),(1,2)\}}$ are dual (Theorem 3.11). In section four, we consider $K_{n,2r}$. We show that it cannot be decomposed non-trivially as $B_1 * B_2$ (Corollary 4.2), and that $K_{n,2r} = N_{n,2r}$ (c.f. (1.6), (4.4), Corollary 4.5). We also show that if p is positive definite, then $(\sum x_i^2)^N p$ is convex for sufficiently large N (Theorem 4.6). In section five, we show that (up to $\pm$) $\mathcal{B}_{2,4}$ consists of a one-parameter family of blenders B_τ, $\tau \in [-\frac{1}{3}, 0]$, where $\tau = \inf\{\lambda : x^4 + 6\lambda x^2y^2 + y^4 \in B_\tau\}$, increasing from $Q_{2,4} = B_0$ to $P_{2,4} = B_{-\frac{1}{3}}$, and that $B_\tau^* = B_{U(\tau)}$, where $U(\tau) = -\frac{1+3\tau}{3-3\tau}$ (Theorem 5.7). In section six, we review the results of $K_{2,4}$ and $K_{2,6}$ in [8, 9, 16] by Dmitriev and the author, and give some new examples in $\partial(K_{2,2r})$. The full analysis of $\mathcal{E}(K_{2,2r})$ seems intractable for $r \geq 4$. Finally, in section seven, we look at sums of 4th powers of binary forms. Conjecture 7.1 states that $p \in W_{2,(u,4)}$ if and only if $p = f^2 + g^2$, where $f, g \in P_{n,2u}$. We show that this is true for $u = 1$ and for even symmetric octics p (Theorems 7.3, 7.4). Our classification of even symmetric octics implies that

$$x^8 + \alpha x^4y^4 + y^8 \in W_{2,(2,4)} \iff \alpha \geq -\tfrac{14}{9}. \tag{1.19}$$

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2. The inner product

For p and q in $F_{n,d}$, we define an inner product with deep roots in 19th century algebraic geometry and analysis. Let

$$[p, q] = \sum_{i \in \mathcal{I}(n,d)} c(i)a(p; i)a(q; i). \tag{2.1}$$

This is the usual Euclidean inner product, if $p \leftrightarrow (c(i)^{1/2}a(p; i)) \in \mathbf{R}^N$. The many properties of this inner product (see Props. 2.1, 2.6 and 2.9) strongly suggest that this is the "correct" inner product for $F_{n,d}$. We present without proof the following observations about the inner product.

Proposition 2.1. [18, pp.2,3]
(i) $[p, q] = [q, p]$.
(ii) $j \in \mathcal{I}(n, d) \implies [p, x^j] = a[p; j]$.
(iii) $\alpha \in \mathbf{R}^n \implies [p, (\alpha\cdot)^d] = p(\alpha)$.
(iv) If $p_m \to p$, *then* $[p_m, q] \to [p, q]$ *for every* $q \in F_{n,d}$.
(v) In particular, taking $q = (u\cdot)^d$, $p_m \to p \implies p_m(u) \to p(u)$ *for all* $u \in \mathbf{R}^n$.

The orthogonal complement of a subspace U of $F_{n,d}$,

$$U^\perp = \{v \in F_{n,d} : [u, v] = 0 \quad \text{for all} \quad u \in U\}, \tag{2.2}$$

is also a subspace of $F_{n,d}$ and $(U^\perp)^\perp = U$. The following result is widely-known and has been frequently proved over the last century, see e.g.[18, p.30].

Proposition 2.2. [18, p.93] *Suppose $S \subset \mathbf{R}^n$ has non-empty interior. Then $F_{n,d}$ is spanned by $\{(\alpha\cdot)^d : \alpha \in S\}$.*

Proof. Let U be the subspace of $F_{n,d}$ spanned by $\{(\alpha\cdot)^d : \alpha \in S\}$ and suppose $q \in U^\perp$. Then $0 = [q, (\alpha\cdot)^d] = q(\alpha)$ for all $\alpha \in S$. Since q is a polynomial which vanishes on an open set, $q = 0$. Thus, $U^\perp = \{0\}$, so $U = (U^\perp)^\perp = \{0\}^\perp = F_{n,d}$. □

Proposition 2.3 (Biermann's Theorem)**.** [18, p.31] *The set $\{(i\cdot)^d : i \in \mathcal{I}(n,d)\}$ is a basis for $F_{n,d}$.*

Proof. We note that there are $N(n,d)$ such forms, so it suffices to construct a dual set $\{g_j : j \in \mathcal{I}(n,d)\} \subset F_{n,d}$ so that $[g_j, (i\cdot)^d] = 0$ if $j \neq i$ and $[g_i, (i\cdot)^d] > 0$. Let

$$g_j(x_1, \dots, x_n) = \prod_{k=1}^{n} \prod_{\ell=0}^{j_k - 1} (dx_k - \ell(x_1 + \cdots + x_n)). \tag{2.3}$$

Each g_j is a product of $\sum_k j_k = d$ linear factors, so $g_j \in F_{n,d}$. The (k,ℓ) factor in (2.3) vanishes at any $x = i \in \mathcal{I}(n,d)$ for which $i_k = \ell$. Thus, $[g_j, (i\cdot)^d] = g_j(i) = 0$ if $i_k \le j_k - 1$ for any k. Since $\sum_k i_k = \sum_k j_k$, it follows that $g_j(i) = 0$ if $j \neq i$. A computation shows that $g_i(i) = d^d \prod_k (i_k!) = d^d d!/c(i)$. □

Prop. 2.3 implies Prop. 2.2 directly, upon mapping $\mathcal{I}(n,d)$ linearly into S.

Proposition 2.4. [18, p.141] *Suppose $B \in \mathcal{B}_{n,d}$ and there exist $p, q \in B$ and $u, v \in \mathbf{R}^n$ so that $p(u) > 0 > q(v)$. Then $B = F_{n,d}$.*

Proof. By Lemma 1.1(iii), $\pm(\alpha\cdot)^d \in B$ for $\alpha \in \mathbf{R}^n$, so by Prop. 2.2, $F_{n,d} \subseteq B$. □

This is the argument Ellison used in [10, p.667] to show that every $p \in F_{n,u(2v+1)}$ is a sum of $(2v+1)$-st powers of $h_k \in F_{n,u}$.

Let $-B = \{-h : h \in B\}$. It is easy to check that if B is a blender, then so is $-B$.

Proposition 2.5. [18, p.141] *If $B \neq \{0\}, F_{n,d}$ is a blender, then $d = 2r$ is even and for a suitable choice of sign, $Q_{n,2r} \subseteq \pm B \subseteq P_{n,2r}$.*

Proof. If $B \neq \{0\}$, then there exists $p \in B$ and $a \in \mathbf{R}^n$ so that $p(a) \neq 0$. If d is odd, then $p(-a) = -p(a)$, and by Prop. 2.4, $B = F_{n,d}$. If d is even, by taking $-B$ if necessary, we may assume that $p(a) \ge 0$. Thus, if $B \neq F_{n,2r}$, then $\pm B \subseteq P_{n,2r}$. On the other hand, Lemma 1.1 and (P1) imply that $Q_{n,2r} \subseteq \pm B$. □

Since $Q_{n,2} = P_{n,2}$, there are no "interesting" blenders of quadratic forms.

The inner product has a useful contravariant property.

Proposition 2.6. [18, p.32] *Suppose $p,\ q \in F_{n,d}$ and $M \in Mat_n(\mathbf{R})$. Then*

$$[p \circ M, q] = [p, q \circ M^t]. \tag{2.4}$$

Proof. By Prop. 2.2, it suffices to prove (2.4) for d-th powers; note that $[p\circ M, q] = [(\alpha M\cdot)^d, (\beta\cdot)^d] = (\alpha M\beta^t)^d = (\alpha(\beta M^t)^t)^d = [(\alpha\cdot)^d, (\beta M^t\cdot)^d] = [p, q\circ M^t]$. ☐

Proposition 2.7. [18, p.46] *If B is a blender, then so is its dual cone B^*.*

Proof. The dual of a closed convex cone is a closed convex cone, so (P1) and (P2) are automatic. Suppose $p\in B, q\in B^*$ and $M\in Mat_n(\mathbf{R})$. Since $p\circ M^t\in B$, we have

$$[p, q\circ M] = [q\circ M, p] = [q, p\circ M^t] = [p\circ M^t, q] \geq 0, \tag{2.5}$$

and so $q\circ M\in B^*$. This verifies (P3). ☐

For $i\in\mathcal{I}(n,d)$, let $D^i = \prod(\frac{\partial}{\partial x_k})^{i_k}$; let $f(D) = \sum c(i)a(f;i)D^i$ be the d-th order differential operator associated to $f\in F_{n,d}$. Since $\frac{\partial}{\partial x_k}$ and $\frac{\partial}{\partial x_\ell}$ commute, $D^iD^j = D^{i+j} = D^jD^i$ for any $i\in\mathcal{I}(n,d)$ and $j\in\mathcal{I}(n,e)$. By multilinearity, $(fg)(D) = f(D)g(D) = g(D)f(D)$ for forms f and g of any degree.

Proposition 2.8. [20, p.183] *If $i,j\in\mathcal{I}(n,d)$ and $i\neq j$, then $D^i(x^j) = 0$ and $D^ix^i = \prod_k(i_k)! = d!/c(i)$.*

Proof. We have

$$D^i(x^j) = \prod_{k=1}^{n}\left(\frac{\partial^{i_k}}{\partial x_k^{i_k}}\right)\prod_{k=1}^{n} x_k^{j_k} = \prod_{k=1}^{n}\frac{\partial^{i_k}(x_k^{j_k})}{\partial x_k^{i_k}}. \tag{2.6}$$

If $i_k > j_k$, then the k-th factor above is zero. If $i\neq j$, then this will happen for at least one k. Otherwise, $i=j$, and the k-th factor is $i_k!$. ☐

We now connect the inner product with differential operators.

Proposition 2.9. [20, p.184]
(i) If $p,q\in F_{n,d}$, then $p(D)q = q(D)p = d![p,q]$.
(ii) If $p, hf\in F_{n,d}$, where $f\in F_{n,k}$ and $h\in F_{n,d-k}$, then

$$d![p,hf] = (d-k)![h, f(D)p]. \tag{2.7}$$

Proof. For (i), we have by Prop. 2.8:

$$\begin{gathered} p(D)q = \sum_{i\in\mathcal{I}(n,d)} c(i)a(p;i)D^i\left(\sum_{j\in\mathcal{I}(n,d)} c(j)a(q;j)x^j\right) = \\ \sum_{i\in\mathcal{I}(n,d)}\sum_{j\in\mathcal{I}(n,d)} c(i)c(j)a(p;i)a(q;j)D^ix^j = \sum_{i\in\mathcal{I}(n,d)} c(i)c(i)a(p;i)a(q;i)D^ix^i \\ = \sum_{i\in\mathcal{I}(n,d)} c(i)^2a(p;i)a(q;i)\frac{d!}{c(i)} = d![p,q] = d![q,p] = q(D)p. \end{gathered} \tag{2.8}$$

(ii) Two applications of (i) give

$$d![p,hf] = (hf)(D)p = h(D)f(D)p = h(D)(f(D)p) = (d-k)![h, f(D)p]. \tag{2.9}$$

☐

Corollary 2.10. *If $p \in F_{n,2r}$, then $Hes(p;u,v) = 2r(2r-1)[p,(u\cdot)^{2r-2}(v\cdot)^2]$.*

Proof. Apply Prop. 2.9 with $h = (u\cdot)^{2r-2}$, $f = (v\cdot)^2$, $d = 2r$ and $k = 2$. We have

$$f(x_1,\dots,x_n) = (v_1x_1 + \cdots + v_nx_n)^2 \implies f(D) = \sum_{i=1}^{n}\sum_{j=1}^{n} v_iv_j\frac{\partial^2}{\partial x_i \partial x_j}, \tag{2.10}$$

so that $[h, f(D)p] = Hes(p;u,v)$ by (1.8) and Prop. 2.1(iii). □

3. Convergence and duals

We shall need some tools to prove that certain convex cones are closed. The first one (see [18, p.37]) is an immediate consequence of Prop. 2.2.

Lemma 3.1. *Suppose $S \subset \mathbf{R}^n$ is bounded and has non-empty interior. Then for $i \in \mathcal{I}(n,d)$ and $p \in F_{n,d}$, $|a(p;i)| \le R_{n,d}(i)\cdot \sup\{|p(x)| : x \in S\}$ for some $R_{n,d}(i)$.*

Proof. Fix $i \in \mathcal{I}(n,d)$. By Prop. 2.2, there exist $\lambda_k(i)$ and $\alpha_k \in S$ so that

$$x^i = \sum_{k=1}^{N(n,d)} \lambda_k(i)(\alpha_k\cdot)^d. \tag{3.1}$$

Taking the inner product of (3.1) with p, we find that

$$a(p;i) = [p,x^i] = \sum_{k=1}^{N(n,d)} \lambda_k(i)[p,(\alpha_k\cdot)^d] = \sum_{k=1}^{N(n,d)} \lambda_k(i)p(\alpha_k). \tag{3.2}$$

Now set $R_{n,d}(i) = \sum_k |\lambda_k(i)|$. □

We define the norm on $F_{n,d}$ in the usual way, by

$$||p||^2 = [p,p] = \sum_{i\in\mathcal{I}(n,d)|} c(i)a(p;i)^2. \tag{3.3}$$

Given a sequence $(p_m) \in F_{n,d}$, the statement that $(|a(p_m;i)|)$ is uniformly bounded for all (i,m) is equivalent to the statement that $(||p_m||)$ is bounded.

Lemma 3.2. *Suppose $(p_{m,r}) \subset F_{n,d}$, $1 \le r \le N$, and suppose that for all (m,r), $|p_{m,r}(u)| \le M$ for $u \in S$, where S is bounded and has non-empty interior. Then there exist $p_r \in F_{n,d}$ and $m_k \to \infty$ so that simultaneously for each r, $p_{m_k,r} \to p_r$.*

Proof. Identify each $p_{m,r}$ with the vector $(a(p_{m,r};i)) \in \mathbf{R}^{N(n,d)}$; these vectors are uniformly bounded by Lemma 3.1. Concatenate them to form a vector $v_m \in \mathbf{R}^{N*N(n,d)}$. By Bolzano-Weierstrass, there is a convergent subsequence (v_{m_k}). The corresponding subsequences of forms are then convergent. □

Even when (p_m) is unbounded, one can still find an interesting subsequence.

Lemma 3.3. *Suppose $(p_m) \subset F_{n,d}$ and $||p_m||$ is unbounded. Then there exists a subsequence p_{m_k} and $\tau_k \to \infty$ so that $\tau_k^{-1}p_{m_k} \to p$, where $p \neq 0$.*

Proof. Let $\mu_m = \max\{|a(p_m;i)|\}$; by hypothesis, (μ_m) is unbounded. Take a subsequence on which $\mu_m \to \infty$ and drop the subscripts. Let $\bar{p}_m = \mu_m^{-1} p_m$. Then each $\bar{p}_m$ has at least one coefficient $a(\bar{p}_m; i(m)) = \pm 1$. Since $\mathcal{I}(n,d)$ is finite, there exists i_0 so that there is a subsequence on which $a(\bar{p}_{m_k}; i_0) = \pm 1$. Taking $-p_{m_k}$ if necessary and dropping the subscripts, we have $a(\bar{p}_m; i_0) = 1$ and $|a(\bar{p}_m; i)| \le 1$ for all (m,i). By Lemma 3.2, $(\bar{p}_m)$ has a convergent subsequence $\bar{p}_{m_k} \to p$, and $a(p; i_0) = 1$, so $p \neq 0$. Since $\bar{p}_{m_k} = \mu_{m_k}^{-1} p_{m_k}$, this is the desired subsequence. $\square$

We state without proof a direct implementation of Carathéodory's Theorem (see e.g. [18, p.27].). It is worth noting that in 1888 (when Carathéodory was 15), Hilbert [12] used this argument with $N(3,6) = 28$ to show that $\Sigma_{3,6}$ is closed.

Proposition 3.4 (Carathéodory's Theorem). *If $r > N(n,d)$, and $h_k \in F_{n,d}$, then there exist $\lambda_k \ge 0$ so that*

$$\sum_{k=1}^{r} h_k = \sum_{k=1}^{N(n,d)} \lambda_k h_{n_k}. \tag{3.4}$$

We use these lemmas to show that if B_1 and B_2 are blenders, then so are $B_1 + B_2$ (c.f. (1.14)) and $B_1 * B_2$ (c.f. (1.16)). We may assume $B_i \neq 0$.

Theorem 3.5.
(i) If $B_i \in \mathcal{B}_{n,2r}$, then $B_1 + B_2 \in \mathcal{B}_{n,2r}$.
(ii) If $B_i \in \mathcal{B}_{n,2r_i}$ and $r = r_1 + r_2$, then $B_1 * B_2 \in \mathcal{B}_{n,2r}$.

Proof. In each case, (P1) is automatic, and since $(p_1 + p_2) \circ M = p_1 \circ M + p_2 \circ M$ and $(p_1 p_2) \circ M = (p_1 \circ M)(p_2 \circ M)$, (P3) is verified. The issue is (P2).

Suppose $B_i \in \mathcal{B}_{n,2r}$ have opposite "sign", say $B_1 \subset P_{n,2r}$ and $B_2 \subset -P_{n,2r}$. Then Prop. 2.4 implies that $B_1 + B_2 = F_{n,2r}$. Otherwise, we may assume that $B_i \subset P_{n,2r_i}$. Suppose $p_{i,m} \in B_i$ and $p_{1,m} + p_{2,m} = p_m \to p$. Let S be the unit ball in $\mathbf{R}^n$. If $\sup\{p(u) : u \in S\} = T$, then for $m \ge m_0$, $\sup\{p_m(u) : u \in S\} \le T + 1$, and since $p_{i,m}$ is psd, it follows that $\sup\{p_{i,m}(u) : u \in S\} \le T + 1$ as well. By Lemma 3.2, there is a common subsequence so that $p_{i,m_k} \to p_i \in B_i$, hence $p = \lim p_{m_k} = p_1 + p_2 \in B_1 + B_2$.

Suppose now $B_i \in \mathcal{B}_{n,2r_i}$, and by taking $\pm B_i$, assume $B_i \subset P_{n,2r_i}$. By Prop. 3.4, a sum such as (1.16) can be compressed into one in which $s \le N(n,2r)$. Write

$$p_m = \sum_{k=1}^{N(n,2r)} p_{1,k,m} p_{2,k,m}, \qquad p_{i,k,m} \in B_i, \tag{3.5}$$

and suppose $p_m \to p$. As above, since p is bounded on S, so is the sequence (p_m), and since each $p_{i,k,m}$ is psd, it follows that the sequence $(p_{1,k,m} p_{2,k,m})$ is bounded on S, and hence by Lemma 3.2, a subsequence of $(p_{1,k,m} p_{2,k,m}) \to p_k$ for some $p_k \in P_{n,2r}$. We need to show that p_k can be written as a product $q_{1,k} q_{2,k}$, where $q_{i,k} \in B_i$. A complication is that the given sequence of factors might not both converge (e.g. if $p_{1,k,m} = m q_{1,k}$ and $p_{2,k,m} = m^{-1} q_{2,k}$), so we need to normalize.

First observe that if $p_k = 0$, we are done. Otherwise, choose $v \in \mathbf{R}^n$ so that $p_k(v) = 1$. Since $p_{1,k,m}(v)p_{2,k,m}(v) \to 1$, $p_{1,k,m}(v)p_{2,k,m}(v) > 0$ for $m \ge m_0$. Drop the first m_0 terms and define

$$(3.6) \qquad q_{1,k,m}(x) = \frac{p_{1,k,m}(x)}{p_{1,k,m}(v)} \in B_1, \qquad q_{2,k,m}(x) = \frac{p_{2,k,m}(x)}{p_{2,k,m}(v)} \in B_2.$$

Then $(q_{1,k,m}q_{2,k,m}) \to p_k$ and $q_{i,k,m}(v) = 1$.

If each $(||q_{i,k,m}||)$ is bounded, then by Lemma 3.2, there are convergent subsequences $q_{i,k,m} \to q_{i,k} \in B_i$ and $p_k = q_{1,k}q_{2,k}$ as desired.

Suppose $(||q_{1,k,m}||)$ is unbounded and $(||q_{2,k,m}||)$ is bounded. Taking the common convergent subsequences from Lemmas 3.2 and 3.3, and dropping subscripts, we have $\tau_m \to \infty$ and $q_{1,k,m} = \tau_m \bar{q}_{1,k,m}$ so that $\bar{q}_{1,k,m} \to \bar{q}_{1,k} \in B_1$ (where $\bar{q}_{1,k} \ne 0$) and $q_{2,k,m} \to q_{2,k} \in B_2$, where $q_{2,k}(v) = \lim q_{2,k,m}(v) = 1$, so $q_{2,k} \ne 0$. But now

$$(3.7) \qquad 0 = \lim_{m\to\infty} \tau_m^{-1} q_{1,k,m}q_{2,k,m} = \lim_{m\to\infty} \bar{q}_{1,k,m}q_{2,k,m} = \bar{q}_1 q_2,$$

a contradiction. If both $(||q_{i,k,m}||)$'s are unbounded, we can write $q_{2,k,m} = \nu_m \bar{q}_{2,k,m}$ with $\nu_m \to \infty$ and $\bar{q}_{2,k,m} \to \bar{q}_{2,k} \ne 0$ and derive a similar contradiction. It follows that the first case holds for each k and so $B_1 * B_2$ satisfies (P2). □

The following theorem was announced in [18, p.47], but the proof was not given.

Theorem 3.6. *If $uv = r$, then $W_{n,(u,2v)}$ is a blender.*

Proof. As we have already seen, (P1) and (P3) are immediate. Suppose $p_m \in W_{n,(u,2v)}$ and $p_m \to p$. Prop. 3.4 says that we can write

$$(3.8) \qquad p_m = \sum_{k=1}^{N(n,2r)} h_{k,m}^{2v}, \qquad h_{k,m} \in F_{n,u}.$$

As before, p (and so (p_m)) is bounded on S, and the summands are psd so $(h_{k,m}^{2v})$ and thus also $(|h_{k,m}|) = ((h_{k,m}^{2v})^{1/(2v)})$ are bounded on S. Taking a convergent subsequence, suppose $(h_{k,m}) \to h_k$. Then $(h_{k,m}^{2v}) \to h_k^{2v}$. Taking a common subsequence for each of the $N(n,2r)$ summands, we see that $p \in W_{n,(u,2v)}$. □

In particular, $\Sigma_{n,2r}$ and $Q_{n,2r}$ are blenders; see [18, p.46].

Theorem 3.7. *If $\sum_i u_i v_i = 2r$, then $W_{n,\{(u_1,2v_1),\dots,(u_m,2v_m)\}} \in \mathcal{B}_{n,2r}$.*

Proof. Note that $W_{n,\{(u_1,2v_1),\dots,(u_m,2v_m)\}} = W_{n,(u_1,2v_1)} * \cdots * W_{n,(u_m,2v_m)}$. □

Proposition 3.8. [18, p.38] *$P_{n,2r}$ and $Q_{n,2r}$ are dual blenders.*

Proof. We have $p \in Q_{n,2r}^*$ if and only if $p \in F_{n,2r}$ and, whenever $\lambda_k \ge 0$ and $\alpha_k \in \mathbf{R}^n$,

$$(3.9) \qquad 0 \le \left[p, \sum_{k=1}^{r} \lambda_k (\alpha_k \cdot)^{2r} \right] = \sum_{k=1}^{r} \lambda_k p(\alpha_k).$$

This is true iff $p(\alpha) \ge 0$ for all $\alpha \in \mathbf{R}^n$; that is, iff $p \in P_{n,2r}$. □

It was a commonplace by the time of [12] that $P_{n,2r} = \Sigma_{n,2r}$ when $n = 2$ or $2r = 2$. Hilbert proved there that $P_{3,4} = \Sigma_{3,4}$ and that strict inclusion is true for other $(n, 2r)$ (see [21].) We say that $p \in P_{n,2r}$ is *positive definite* or *pd* if $p(u) = 0$ only for $u = 0$. It follows that $p \in int(P_{n,2r})$ if and only if p is pd.

Blenders are cousins of orbitopes. An *orbitope* is the convex hull of an orbit of a compact algebraic group G acting linearly on a real vector space; see [24, p.1]. The key differences from blenders are that it is a single orbit, and that G is compact. One object which is both a blender and an orbitope is $Q_{n,2r}$, which is named $\mathcal{V}_{n,2r}$ (and called the *Veronese orbitope*) in [24].

The duals of the Waring blenders can be explicitly given.

Proposition 3.9. [18, p.47] *Given $p \in F_{n,2uv}$, define the form $H_p(t) \in F_{N(n,u),2v}$, in variables $\{t(\ell)\}$ indexed by $\{\ell \in \mathcal{I}(n,u)\}$, by*

$$H_p(\{t(\ell_j)\}) = \sum_{\ell_1 \in \mathcal{I}(n,u)} \cdots \sum_{\ell_{2v} \in \mathcal{I}(n,u)} a(p; \ell_1 + \cdots + \ell_{2v}) t(\ell_1) \cdots t(\ell_{2v}). \tag{3.10}$$

Then $p \in W^*_{n,(u,2v)}$ if and only if $H_p \in P_{N(n,u),2v}$.

Proof. We have $p \in W^*_{n,(u,v)}$ if and only if, for every form $g \in F_{n,u}$, $[p, g^{2v}] \geq 0$. Writing $g \in F_{n,u}$ with coefficients $\{t(\ell) : \ell \in \mathcal{I}(n,u)\}$, we have:

$$\begin{gathered} g(x) = \sum_{\ell \in \mathcal{I}(n,u)} t(\ell) x^\ell \implies \\ g^{2v}(x) = \sum_{\ell_1 \in \mathcal{I}(n,u)} \cdots \sum_{\ell_{2v} \in \mathcal{I}(n,u)} t(\ell_1) \cdots t(\ell_{2v}) x^{\ell_1 + \cdots + \ell_{2v}}. \end{gathered} \tag{3.11}$$

It follows from (2.1) and (3.10) that $[p, g^v] = H_p(t(\ell))$. $\square$

If $v = 1$, then $\mathcal{I}(n,1) = \{e_i\}$ and, upon writing $t(e_i) = y_i$, $H_p(y_1, \ldots, y_n) = p(y)$; we recover $Q^*_{n,2r} = P_{n,2r}$. If $u = 1$, then H_p becomes the classical catalecticant and

$$p \in \Sigma^*_{2,2r} \iff H_p(t) = \sum_{i \in \mathcal{I}(n,r)} \sum_{j \in \mathcal{I}(n,r)} a(p; i+j) t(\ell_i) t(\ell_j) \quad \text{is } psd. \tag{3.12}$$

This shows that $\Sigma_{n,2r}$ is a spectrahedron (see [24, p.27]).

Theorem 3.10. *If $\sum v_i = r$, then $W_{2,\{(1,2v_1),\ldots,(1,2v_m)\}} = P_{2,2r}$ iff $m = r$ and $v_i = 1$.*

Proof. If $p \in P_{2,2r} = \Sigma_{2,2r}$, then $p = f_1^2 + f_2^2$, where $f_i \in F_{2,r}$. Factor $\pm f_i$ into a product of linear and pd quadratic factors (themselves a sum of two squares):

$$f_i = \prod_j \ell_{1,j} \prod_k (\ell_{2,k}^2 + \ell_{3,k}^2). \tag{3.13}$$

Then, using (1.18) and expanding the product below, we see that

$$f_i^2 = \prod_j \ell_{1,j}^2 \prod_k \left((\ell_{2,k}^2 - \ell_{3,k}^2)^2 + (2\ell_{2,k}\ell_{3,k})^2 \right) \in W_{2,\{(1,2),\ldots,(1,2)\}}. \tag{3.14}$$

The converse inclusion follows from Prop. 2.5.

Suppose $m < r$ and suppose

$$\prod_{\ell=1}^{r}(x-\ell y)^2 = \sum_{k=1}^{s} h_{k,1}^{2v_1}\cdots h_{k,m}^{2v_m}, \quad h_{k,i}(x,y) = \alpha_{k,i}x + \beta_{k,i}y \in F_{2,1}. \tag{3.15}$$

Then for each k, we have

$$\prod_{\ell=1}^{r}(x-\ell y) \ \Big| \ \prod_{i=1}^{m}(\alpha_{k,i}x + \beta_{k,i}y); \tag{3.16}$$

since $m < r$, the right-hand side is 0, and we have a contradiction. $\square$

Finally, we have a simple expression for $K_{n,2r}^*$; this seems to be implicit in [2].

Theorem 3.11. *$K_{n,2r}$ and $W_{n,\{(1,2r-2),(1,2)\}}$ are dual blenders.*

Proof. By Corollary 2.10 and the Hessian definition, p is convex if and only if $0 \le Hes(p;u,v) = 2r(2r-1)[p,(u\cdot)^{2r-2}(v\cdot)^2]$ for all $u, v \in \mathbf{R}^n$. $\square$

It follows from Theorems 3.10 and 3.11 that $K_{2,4}^* = W_{2,\{(1,2),(1,2)\}} = P_{2,4}$, so $K_{2,4} = Q_{2,4}$. For $r \ge 3$, $K_{2,2r}^* = W_{2,\{(1,2r-2),(1,2)\}} \subsetneq P_{2,4}$, so $K_{2,2r} \supsetneq Q_{2,2r}$. We return to this topic in section six.

4. $K_{n,2r}$: convex forms

In this section, we prove some general results for $K_{n,2r}$. Since $p \in K_{n,2r}$ if and only if $Hes(p;u,v)$ is psd and $Hes(p;u,u) = 2r(2r-1)p(u)$, we get an alternative proof that $K_{2,2r} \subseteq P_{n,2r}$. We also know from Theorem 3.11 that $p \in int(K_{n,2r})$ if and only if $[p,q] > 0$ for $0 \ne q \in W_{n,\{(1,2r-2),(1,2)\}}$; accordingly, $int(K_{n,2r})$ is the set of $p \in K_{2,2r}$ so that $Hes(p;u,v)$ is positive definite as a bihomogeneous form in the variables $u \in \mathbf{R}^n$ and $v \in \mathbf{R}^n$. Equivalently, $p \in K_{n.2r}$ is in $\partial(K_{n,2r})$ if and only if there exist $u_0 \ne 0, v_0 \ne 0$ such that $Hes(p;u_0,v_0) = 0$.

Although the psd and sos properties are preserved under homogenization and dehomogenization, this is not true for convexity. For example, t^2-1 is a convex polynomial which cannot be homogenized to a convex form, because it is not definite. As a pd polynomial in one variable, t^4+12t^2+1 is convex, but if $p(x,y) = x^4+12x^2y^2+y^4$, then $Hes(p;(1,1),(v_1,v_2)) = 36v_1^2+96v_1v_2+36v_2^2$ is not psd, so p is not convex.

Proposition 4.1. *If $p \in K_{2,2r}$, then there is a pd form q in $\le n$ variables and $\bar p \sim p$ such that $\bar p(x) = q(x_k,\dots,x_n)$.*

Proof. If p is pd, there is nothing to prove. Otherwise, we can assume that $p \sim \bar p$, where $\bar p$ is convex and $\bar p(e_1) = 0$. We shall show that $\bar p = \bar p(x_2,\dots,x_n)$. Repeated application of this argument then proves the result.

Suppose otherwise that x_1 appears in a term of $\bar p$ and let $m \ge 1$ be the largest such power of x_1; write the associated terms in $\bar p$ as $x_1^m h(x_2,...,x_n)$. After an additional invertible linear change involving $(x_2, \dots, x_n)$, we may assume that one of these terms is $x_1^m x_2^{2r-m}$. We then have

$$\bar{p}(x_1, x_2, 0, \dots, 0) = x_1^m x_2^{2r-m} + \text{lower order terms in } x_1 \tag{4.1}$$

which implies that

$$\frac{\partial^2 \bar{p}}{\partial x_1^2}\frac{\partial^2 \bar{p}}{\partial x_2^2} - \left(\frac{\partial^2 \bar{p}}{\partial x_1 \partial x_2}\right)^2 = \tag{4.2}$$
$$-(2r-1)m(2r-m)x_1^{2m-2}x_2^{4r-2m-2} + \text{lower order terms in } x_1.$$

Since $r \geq 1$ and $1 \leq m \leq 2r-1$, (4.2) cannot be psd, and this contradiction shows that x_1 does not occur in $\bar{p}$. □

Corollary 4.2. *There do not exist $B_i \in \mathcal{B}_{n,2r_i}$, $r_i \geq 1$, so that $K_{n,2r_1+2r_2} = B_1 * B_2$.*

Proof. It follows from Prop. 2.5 that $x_i^{2r_i} \in B_i$, hence $x_1^{2r_1}x_2^{2r_2} \in B_1 * B_2$, but by Prop. 4.1, this form is not convex. □

The next theorem connects $K_{n,2r}$ with the blender $N_{n,2r}$ defined in [18, p.119-120]. Let $E = < e_1, \dots, e_n >$ be a real n-dimensional vector space. We say that f is a *norm-function* on E if, after defining

$$||x_1e_1 + \cdots + x_ne_n|| = f(x_1, \dots, x_n), \tag{4.3}$$

the pair $(E, ||\cdot||)$ is a Banach space. Let

$$N_{n,d} := \{p \in F_{n,d} : p^{1/d} \text{ is a norm function}\}. \tag{4.4}$$

A necessary condition is that $f = p^{1/d} \geq 0$, hence $d = 2r$ is even and $p \in P_{n,2r}$. For example, if $p(x) = \sum_k x_k^2$, then (4.3) with $f = p^{1/2}$ gives $\mathbf{R}^n$ with the Euclidean norm. If $(E, ||\cdot||)$ is isometric to a subspace of some $L_{2r}(X, \mu)$, then $f^{2r} \in Q_{n,2r}$. The following theorem was proved in the author's thesis; see [15, 16].

Proposition 4.3. [16, Thm.1] *If $p \in P_{n,2r}$, then $p \in N_{n,2r}$ iff for all $u, v \in \mathbf{R}^n$, $p(u_1 + tv_1, \dots, u_n + tv_n)^{1/d}$ is a convex function of t.*

It is not obvious that $N_{n,2r}$ is a blender; in fact, $N_{n,2r} = K_{n,2r}$! The connection is a proposition whose provenance is unclear. It appears in Rockafellar's monograph [23, Cor.15.3.1], where it is attributed to Lorch [13], although the derivation is not transparent. V. I. Dmitriev (see section 6) attributes the result to an observation by his advisor S. G. Krein in 1969. Note below that q is *not* homogeneous.

Proposition 4.4. *Suppose $p \in P_{n,2r}$ and $p(1, 0, ..., 0) > 0$. Let*

$$q(x_2, \dots, x_n) = p(1, x_2, \dots, x_n). \tag{4.5}$$

Then $p \in K_{n,2r}$ if and only if $q^{1/(2r)}(x_2, \dots, x_n)$ is convex.

Corollary 4.5. $K_{n,2r} = N_{n,2r}$.

Proof of Prop. 4.4. A function is convex if and only if it is convex when restricted to all two-dimensional subspaces. Consider all $a \in \mathbf{R}^N$ with $a_1 = 1$. Suppose we can show that $Hes(p; a, u)$ is psd in u if and only if $q^{1/(2r)}$ is convex at $(a_2, \dots, a_n)$. By homogeneity, this occurs if and only if $Hes(p; a, u)$ is psd in u for every a with $a_1 \neq 0$ and by continuity, this holds if and only if $Hes(p; a, u)$ is psd for all a, u. Thus, it suffices to set $a_1 = 1$ and prove the equivalence pointwise.

Fix $(a_2, \dots, a_n)$ and let

$$\begin{aligned} \tilde{p}(x_1, x_2 \dots, x_n) &= p(x_1, x_2 + a_2 x_1, \dots, x_n + a_n x_1), \\ \tilde{q}(x_2, \dots, x_n) &= \tilde{p}(1, x_2, \dots x_n) = q(x_2 + a_2, \dots, x_n + a_n) \end{aligned} \tag{4.6}$$

Then p and $q^{1/(2r)}$ are convex at a and $(a_2, \dots, a_n)$ iff $\tilde{p}$ and $\tilde{q}$ are convex at e_1 and 0, and we can drop the tildes and assume that $a_k = 0$ for $k \geq 2$, so $a = e_1$. Since it suffices to look at all two-dimensional subspaces containing e_1, we make one more change of variables in $(x_2, \dots, x_n)$, and assume this subspace is $\{(x_1, x_2, 0, \dots, 0)\}$.

Suppose now that

$$h(x_1, x_2) = p(x_1, x_2, 0, \dots, 0) = a_0 x_1^{2r} + \binom{2r}{1} a_1 x_1^{2r-1} x_2 + \binom{2r}{2} a_2 x_1^{2r-2} x_2^2 + \dots . \tag{4.7}$$

Then

$$Hes(h; (1,0), (v_1, v_2)) = 2r(2r-1)(a_0 v_1^2 + 2a_1 v_1 v_2 + a_2 v_2^2), \tag{4.8}$$

and since $a_0 = p(e_1) > 0$, this is psd iff $a_0 a_2 \geq a_1^2$. On the other hand,

$$q(t) = p(1, t) = a_0 + \binom{2r}{1} a_1 t + \binom{2r}{2} a_2 t^2 + \dots \tag{4.9}$$

and a routine computation shows that

$$(q^{(1/(2r))})''(0) = (2r-1) a_0^{-2+1/(2r)} (a_0 a_2 - a_1^2). \tag{4.10}$$

Thus the two conditions hold simultanously. $\square$

A more complicated proof computes the Hessian of p, uses the Euler PDE ($2rp = \sum x_i \frac{\partial p}{\partial x_i}$ and $(2r-1)\frac{\partial p}{\partial x_i} = \sum x_j \frac{\partial^2 p}{\partial x_i \partial x_j}$) to replace partials involving x_1 with partials involving only the other variables. The discriminant of this Hessian with respect to u_1 (after a change of variables) becomes a positive multiple of the Hessian of $q^{1/(2r)}$.

We conclude this section with a peculiar result which implies that every pd form is, in a computable way, the restriction of a convex form on S^{n-1}.

Theorem 4.6. *Suppose $p \in P_{n,2r}$ is pd, and let $p_N := (\sum_j x_j^2)^N p$. Then there exists N so that $p_N \in K_{n,2r+2N}$.*

Proof. Since p is pd, it is bounded away from 0 on S^{n-1} and so there are uniform upper bounds T for $|p(x)^{-1} \nabla_u(p)(x)|$ and U for $|p(x)^{-1} \nabla_u^2(p)(x)|$, for $x, u \in S^{n-1}$. Since $\sum x_i^2$ is rotation-invariant, once again it suffices to show that p_N is convex at $(1, 0, \dots, 0)$, given $x_3 = \dots = x_n = 0$. We claim that if $N > (T^2 + U)/2$, then p_N

is convex. By Prop. 4.4, it suffices to show that $p_N^{1/(2N+2r)}(1,t,0,\dots,0)$ is convex at $t=0$. Writing down the relevant Taylor series, this becomes

$$(1+t^2)^{N/(2N+2r)}(1+\alpha t+\tfrac{1}{2}\beta t^2+\dots)^{1/(2N+2r)}, \tag{4.11}$$

where $|\alpha|\le T$ and $|\beta|\le U$. By expanding the product, a standard computation shows that the second derivative at $t=0$ is

$$\frac{N}{N+r}+\frac{1}{2N+2r}\cdot b-\frac{2N+2r-1}{(2N+2r)^2}\cdot a^2\ge\frac{1}{2N+2r}\left(2N-U-T^2\right)\ge 0. \tag{4.12}$$

□

Greg Blekherman pointed out to the author's chagrin in Banff that Theorem 4.6 follows from [19, Thm.3.12]: if p is pd, then there exists N so that $p_N\in Q_{n,2r+2N}$. This was used in [19] to show that $P_N\in\Sigma_{n,2r+2N}$; it also implies that $p\in K_{n,2r+2N}$. The proof of [19, Thm.3.12] is much less elementary.

We conclude this section with a computational illustration of the proof of Theorem 4.6. If $a>0$, then x^2+ay^2 is convex, but if $r\ge 1$ and $(x^2+y^2)^r(x^2+ax^2)\in K_{2,2r+2}$ for all $a>0$, then by (P2), $x^2(x^2+y^2)^r$ would be convex, violating Prop. 4.1.

Theorem 4.7.

$$(x^2+y^2)^r(x^2+ax^2)\in K_{2,2r+2}\iff a+1/a\le 8r+18+8/r. \tag{4.13}$$

Proof. Let $p(x,y)=(x^2+y^2)^r(x^2+ax^2)$. A computation shows that

$$\begin{aligned}&\frac{\partial^2 p}{\partial x^2}\frac{\partial^2 p}{\partial y^2}-\left(\frac{\partial^2 p}{\partial x\partial y}\right)^2=4(2r+1)(x^2+y^2)^{2r-2}q(x,y),\quad\text{where}\quad q(x,y)=\\&(1+r)(a+r)x^4+(2a-r+6ar-a^2r+2ar^2)x^2y^2+a(1+r)(1+ar)y^4.\end{aligned} \tag{4.14}$$

Another computation shows that

$$\begin{aligned}&4(1+r)(a+r)q(x,y)\\&=(2(1+r)(a+r)x^2+(2a-r+6ar-a^2r+2ar^2)y^2)^2\\&\quad+4ar^2(a-1)^2\big((8r+18+8/r)-(a+1/a)\big)y^4.\end{aligned} \tag{4.15}$$

If $a+1/a\le 8r+18+8/r$, then (4.15) shows that q is psd. Suppose $a+1/a>8r+18+8/r$. Observe that $2a-r+6ar-a^2r+2ar^2\ge 0$ if and only if $(a+1/a)\le 2r+6+2/r$, so in this case, $2a-r+6ar-a^2r+2ar^2<0$ and we can choose $(x,y)=(x_0,y_0)\ne(0,0)$ to make the first square in (4.15) equal to zero. It then follows that $4(1+r)(a+r)q(x_0,y_0)<0$. □

In particular, $(x^2+y^2)(x^2+ay^2)\in K_{2,4}\iff 17-12\sqrt{2}\le a\le 17+12\sqrt{2}$.

5. $\mathcal{B}_{2,4}$: BINARY QUARTIC BLENDERS

In view of Prop. 2.5, the simplest non-trivial opportunity to classify blenders comes with the binary quartics. Throughout this section, we choose a sign for $\pm B \in \mathcal{B}_{2,4}$ and assume that $B \subset P_{2,4}$. We shall show that $\mathcal{B}_{2,4}$ is a one-parameter nested family of blenders increasing from $Q_{2,4}$ to $P_{2,4}$. It is also convenient to let $Z_{2,4}$ denote the set of $p \in P_{2,4}$ which are neither pd not a 4th power; if $p \in Z_{2,4}$, then $p = \ell^2 h$, where ℓ is linear and h is a psd quadratic form relatively prime to ℓ.

Lemma 5.1. *If $B \in \mathcal{B}_{2,4}$ and $0 \neq p \in B \cap Z_{2,4}$, then $B = P_{2,4}$.*

Proof. We have $p \sim q$, where $q(x,y) = x^2(ax^2 + 2bxy + cy^2) \in B$, $ac - b^2 \geq 0$ and $c > 0$. But

$$x^2(ax^2 + 2bxy + cy^2) = x^2\left(\left(\tfrac{ac-b^2}{c}\right)x^2 + c\left(\tfrac{b}{c}x + y\right)^2\right) \sim x^2(dx^2 + cy^2), \tag{5.1}$$

and $d \geq 0$. Next, $(x,y) \mapsto (\epsilon x, \epsilon^{-1}y)$ shows that $\epsilon^2 dx^4 + cx^2y^2 \in B$, so $x^2y^2 \in B$ by (P2) and $\ell_1^2\ell_2^2 \in B$ by (P3). Thus, $W_{2,\{(1,2),(1,2)\}} = P_{2,4} \subseteq B$ by Theorem 3.10. $\square$

This lemma illustrates one difference between blenders and orbitopes. If $G = SO(2)$ and $p(x,y) = x^2(x^2+y^2)$, then the image of p under the action of G will be $\{(\cos tx + \sin ty)^2(x^2+y^2)\}$, so even taking scalar multiples into account, the convex hull will not contain the 4th powers or the square of an indefinite quadratic.

A binary quartic of particular importance is

$$f_\lambda(x,y) := x^4 + 6\lambda x^2y^2 + y^4; \tag{5.2}$$

we also define

$$g_\lambda(x,y) := f_\lambda(x+y, x-y) = (2+6\lambda)x^4 + (12-12\lambda)x^2y^2 + (2+6\lambda)y^4. \tag{5.3}$$

We shall need two special fractional linear transformations. Let

$$T(z) := \frac{1-z}{1+3z}, \qquad U(z) := -\frac{1+3z}{3-3z}. \tag{5.4}$$

It follows from (5.2) that $g_\lambda = (2+6\lambda)f_{T(\lambda)}$, hence for $\lambda \neq -\frac{1}{3}$, $f_\lambda \sim f_{T(\lambda)}$. Note that $T(T(z)) = z$, $T(0) = 1$, $T(\frac{1}{3}) = \frac{1}{3}$, and $T(-\frac{1}{3}) = \infty$ (corresponding to $(x^2-y^2)^2 \sim x^2y^2$); T gives a 1-1 decreasing map between $[\frac{1}{3}, \infty)$ and $(-\frac{1}{3}, \frac{1}{3}]$. We also have

$$[f_\lambda, g_\mu] = (2+6\mu) + \lambda(12-12\mu) + (2+6\mu) = 4(1+3\lambda+3\mu-3\lambda\mu). \tag{5.5}$$

Note that $U(U(z)) = z$, $U(0) = -\frac{1}{3}$, U gives a 1-1 decreasing map from $[-\frac{1}{3}, 0]$ to itself, and

$$[f_\lambda, g_{U(\lambda)+\tau}] = 12(1-\lambda)\tau. \tag{5.6}$$

It follows from (5.6) that $[f_\lambda, g_{U(\lambda)}] = 0$, and if $\lambda < 1$ and $\mu < U(\lambda)$, then $[f_\lambda, g_\mu] < 0$.

It is easy to see directly from (5.2) that f_λ is psd iff $\lambda \in [-\frac{1}{3}, \infty)$, and pd iff $\lambda \in (-\frac{1}{3}, \infty)$, and from (P3) that, if $B \in \mathcal{B}_{2,4}$, then

$$f_\lambda \in B \iff f_{T(\lambda)} \in B. \tag{5.7}$$

By (P1), if $-\frac{1}{3} < \lambda \le \frac{1}{3}$, then $f_\lambda \in B$ implies that $f_\mu \in B$ for $\mu \in [\lambda, T(\lambda)]$.

It is classically known that a "general" binary quartic can be put into the shape f_λ for some λ after an invertible linear transformation. However there is no guarantee that the coefficients of the transformation are real, and the result is not universal: $x^4 \not\sim f_\lambda$. The following first appeared in [14, Thm.6].

Proposition 5.2. *If $p \in P_{2,4}$ is pd, then $p \sim f_\lambda$ for some $\lambda \in (-\frac{1}{3}, \frac{1}{3}]$.*

Proof. Suppose first $p = g^2$. Then g is pd, so $g \sim x^2 + y^2$ and $p \sim f_{\frac{1}{3}}$.

If p is not a perfect square, then it is a product of two pd quadratic forms; we may assume that $p(x,y) = (x^2+y^2)q(x,y)$, with

$$q(x,y) = ax^2 + 2bxy + cy^2. \tag{5.8}$$

A "rotation of axes" fixes x^2+y^2 and takes q into $dx^2 + ey^2$ with $d, e > 0$, $d \ne e$, so $p \sim (x^2+y^2)(dx^2+ey^2)$. Now, $(x,y) \mapsto (d^{-1/4}x, e^{-1/4}y)$ gives $p \sim f_\mu$, where $\mu = \frac{1}{6}(\gamma + \gamma^{-1}) > \frac{1}{3}$ for $\gamma = \sqrt{d/e} \ne 1$. Thus, $p \sim f_{T(\mu)}$ where $T(\mu) \in (-\frac{1}{3}, \frac{1}{3})$. $\square$

We need some results from classical algebraic geometry. Suppose

$$p(x,y) = \sum_{k=0}^{4} \binom{4}{k} a_k(p) x^{4-k} y^k. \tag{5.9}$$

The two "fundamental invariants" of p are

$$\begin{gathered} I(p) = a_0(p)a_4(p) - 4a_1(p)a_3(p) + 3a_2(p)^2, \\ J(p) = \det \begin{vmatrix} a_0(p) & a_1(p) & a_2(p) \\ a_1(p) & a_2(p) & a_3(p) \\ a_2(p) & a_3(p) & a_4(p) \end{vmatrix}. \end{gathered} \tag{5.10}$$

(Note $J(p)$ is the determinant of the catalecticant matrix H_p.) We have $I(f_\lambda) = 1 + 3\lambda^2$ and $J(f_\lambda) = \lambda - \lambda^3$, but $I(x^4) = J(x^4) = 0$. It follows from Prop. 5.2 that if p is pd, then $I(p) > 0$. It is easily checked that if $q(x,y) = p(ax+by, cx+dy)$, then

$$I(q) = (ad-bc)^4 I(p), \qquad J(q) = (ad-bc)^6 J(p). \tag{5.11}$$

Let

$$K(p) := \frac{J(p)}{I(p)^{3/2}}. \tag{5.12}$$

It follows from (5.11) and (5.12) that, if $p \sim q$, then $K(q) = K(p)$. In particular,

$$p \sim f_\lambda \implies K(p) = K(f_\lambda) = \phi(\lambda) := \frac{\lambda - \lambda^3}{(1+3\lambda^2)^{3/2}}. \tag{5.13}$$

Lemma 5.3. *If p is pd, then $p \sim f_\lambda$, where λ is the unique solution in $(-\frac{1}{3}, \frac{1}{3}]$ to $K(p) = \phi(\lambda)$. If $p \in Z_{2,4}$, then $K(p) = \phi(-\frac{1}{3})$.*

Proof. By Proposition 5.2, $p \sim f_\lambda$ for some $\lambda \in (-\frac{1}{3}, \frac{1}{3}]$. A routine computation shows that $f'(\lambda) = (1 - 9\lambda^2)(1 + 3\lambda^2)^{-5/2}$ is positive on $(-\frac{1}{3}, \frac{1}{3})$, hence ϕ is strictly increasing. By Lemma 5.1, if $p \in Z_{2,4}$, then $p \sim q$, where $q(x, y) = dx^4 + 6ex^2y^2$ for some $e > 0$. Since $I(q) = 3e^2$ and $J(q) = -e^3$, $K(p) = K(q) = 3^{-3/2} = \phi(-\frac{1}{3})$. □

Theorem 5.4. *Suppose* $r, s \in [-\frac{1}{3}, 0]$, *and suppose* $1 + 3r + 3s - 3rs = 0$; *that is,* $s = U(r)$. *If* $p \in [[f_r]]$ *and* $q \in [[f_s]]$, *then* $[p, q] \geq 0$.

Proof. Suppose $p = f_r \circ M_1$ and $q = f_s \circ M_2$. Then

$$[p, q] = [f_r \circ M_1, f_s \circ M_2] = [f_r, f_s \circ M_2 M_1^t], \tag{5.14}$$

hence it suffices to show that for all a, b, c, d,

$$\Psi(a, b, c, d; r, s) := [f_r(x, y), f_s(ax + by, cx + dy)] \geq 0 \tag{5.15}$$

A calculation shows that

$$\begin{gathered}\Psi(a, b, c, d; r, s) = a^4 + b^4 + c^4 + d^4+ \\ 6r(a^2b^2 + c^2d^2) + 6s(a^2c^2 + b^2d^2) + 6rs(a^2d^2 + 4abcd + b^2c^2).\end{gathered} \tag{5.16}$$

When $s = U(r)$, a sos expression can be found:

$$\begin{gathered}2(1 - r)\Psi(a, b, c, d; r, U(r)) = (1 + r)(1 + 3r)(a^2 + b^2 - c^2 - d^2)^2 \\ -4r(a^2 + c^2 - b^2 - d^2)^2 + (1 + r)(1 - 3r)(a^2 + d^2 - b^2 - c^2)^2 \\ -8r(1 + 3r)(ab + cd)^2,\end{gathered} \tag{5.17}$$

which is non-negative when $r \in [-\frac{1}{3}, 0]$. Note that $\Psi(1, 1, 1, -1; r, U(r)) = 0$; reaffirming that $[f_r, g_{U(r)}] = 0$. □

Theorem 5.5. *Suppose* $r, s \in [-\frac{1}{3}, 0]$. *If* $s \geq U(r)$, $p \in [[f_r]]$ *and* $q \in [[f_s]]$, *then* $[p, q] \geq 0$. *If* $s < U(r)$, *then there exist* $p \in [[f_r]]$ *and* $q \in [[f_s]]$ *so that* $[p, q] < 0$.

Proof. If $0 \geq s \geq U(r)$, then $s \in [U(r), T(U(r))]$, hence f_s is a convex combination of $f_{U(r)}$ and $f_{T(U(r))}$, and each $f_s \circ M$ is a convex combination of $f_{U(r)} \circ M$ and $f_{T(U(r))} \circ M$. By Theorem 5.4, $[f_r, f_s \circ M]$ is a convex combination of non-negative numbers and is non-negative. If $U(r) \geq s \geq -\frac{1}{3}$, then $[f_r, g_s] < 0$ by (5.6). □

We now have the tools to analyze $B \in \mathcal{B}_{2,4}$. If $Q_{2,4} \subseteq B \subseteq P_{2,4}$, let

$$\Delta(B) = \{\lambda \in \mathbf{R} : f_\lambda \in B\}. \tag{5.18}$$

Theorem 5.6. *If* $B \subset F_{2,4}$ *is a blender, then* $\Delta(B) = [\tau, T(\tau)]$ *for some* $\tau \in [-\frac{1}{3}, 0]$.

Proof. By (P2), $\Delta(B)$ is a closed interval. We have seen that $\Delta(P_{2,4}) = [-\frac{1}{3}, \infty)$. Since $Q_{2,4} = P_{2,4}^* = \Sigma_{2,4}^*$, by (3.12), $f_\lambda \in Q_{2,4}$ if and only if $\begin{pmatrix} 1 & 0 & \lambda \\ 0 & \lambda & 0 \\ \lambda & 0 & 1 \end{pmatrix}$ is psd; that is, $\Delta(Q_{2,4}) = [0, 1]$. Otherwise, let $\tau = \inf\{\lambda : f_\lambda \in B\}$. Since $Q_{2,4} \subsetneq B \subsetneq P_{2,4}$, $\tau \in (-\frac{1}{3}, 0)$. By (P2), $f_\tau \in B$ and by (P3), $f_{T(\tau)} \in B$, and by convexity, $f_\nu \in B$ for $\nu \in [\tau, T(\tau)]$. If $\nu < \tau$, then $f_\nu \notin B$ by definition. If $\nu > T(\tau)$ and $f_\nu \in B$, then $f_{T(\nu)} \in B$ and $T(\nu) < T(T(\tau)) = \tau$, a contradiction. □

Now, for $\tau \in [-\frac{1}{3}, 0]$, let

$$B_\tau := \bigcup_{\tau \le \lambda \le \frac{1}{3}} [[f_\lambda]] = \{p : p \sim f_\lambda, \tau \le \lambda \le \tfrac{1}{3}\} \cup \{(\alpha x + \beta y)^4 : \alpha, \beta \in \mathbf{R}\}. \tag{5.19}$$

Theorem 5.7. *If $B \in \mathcal{B}_{2,4}$, then $B = B_\tau$ for some $\tau \in [-\frac{1}{3}, 0]$ and $B_\tau^* = B_{U(\tau)}$.*

Proof. Suppose B is a blender and $Q_{2,4} \subsetneq B \subsetneq P_{2,4}$. Then $\Delta(B) = [\tau, T(\tau)]$ by Theorem 5.6, so $B = B_\tau$ by Prop. 5.2. We need to show that each such B_τ is a blender. Since $B_0 = Q_{2,4}$ and $B_{-\frac{1}{3}} = P_{2,4}$ are blenders, we may assume $\tau > -\frac{1}{3}$ and all $p \in B_\tau$ are pd. Clearly, (P3) holds in B_τ.

Suppose $p_m \in B_\tau$ and $p_m \to p$. If p is a 4th power, then $p \in B_\tau$. If p is pd, then $K(p_m) \to K(p)$ by (5.11), (5.12) and continuity. In any case, $K(p_m) \ge \phi(\tau)$, so $K(p) \ge \phi(\tau)$ and $p \in B_\tau$. Finally, if $p \in Z_{2,4}$, then $K(p_m) \ge \phi(\tau) > \phi(-\frac{1}{3}) = K(p)$ by Lemma 5.3, and this contradiction completes the proof of (P2).

We turn to (P1). Suppose $p, q \in B_\tau$ and $p + q \notin B_\tau$. Since $p + q$ is pd, $p + q \sim f_\lambda$ for some $\lambda < \tau$, and so there exists M so that $p \circ M + q \circ M = f_\tau$. But now, (5.5) and Theorem 5.5 give a contradiction:

$$0 > [f_\lambda, g_{U(\tau)}] = [p \circ M, g_{U(\tau)}] + [q \circ M, g_{U(\tau)}] \ge 0. \tag{5.20}$$

Thus, $p + q \in B_\tau$ and (P1) is satisfied, showing that B_τ is a blender. It follows from Prop. 2.7 and Theorem 5.5 that $B_\tau^* = B_\nu$ for some ν. But by Theorem 5.5, $B_{U(\tau)} \subseteq B_\tau^*$ and if $\lambda < U(\tau)$, then $f_\lambda \notin B_\nu^*$, thus $B_\tau^* = B_{U(\tau)}$. □

A computation shows that $\phi^2(\lambda) + \phi^2(U(\lambda)) = \frac{1}{27}$, and this gives an alternate way of describing the dual cones. Regrettably, this result was garbled in [18, p.141] into the statement that $B_\tau^* = B_\nu$, where $\tau^2 + \nu^2 = \frac{1}{9}$. The self-dual blender $B_{\nu_0} = B_{\nu_0}^*$ occurs for $\nu_0 = 1 - \sqrt{4/3}$. We know of no other interesting properties of B_{μ_0}.

6. $K_{2,2r}$: BINARY CONVEX FORMS

The author's Ph.D. thesis, submitted in 1976 and published as [15, 16] in 1978 and 1979, discussed $N_{n,2r}$. (The identification of $N_{n,2r}$ and $K_{n,2r}$ was not made there.) Unbeknownst to him, V. I. Dmitriev had earlier worked on similar questions at Kharkov University. In 1969, S. Krein, Dmitriev's advisor, had asked about the extreme elements of $K_{2,2r}$. Dmitriev wrote [8] in 1973 and [9] in 1991. Dmitriev writes in [9]: "I am not aware of any articles on this topic, except [8]." We have seen [9] both in its original Russian and in the English translation. We have not yet seen [8] (although UI Interlibrary Loan is still trying!), and our comments on [9] are based on references in [9]. There are at least two mathematicians named V. I. Dmitriev in MathSciNet; the author of [8, 9] is affiliated with Kursk State Technical University.

Let

$$q_\lambda(x, y) = x^6 + 6\lambda x^5 y + 15\lambda^2 x^4 y^2 + 20\lambda^3 x^3 y^3 + 15\lambda^2 x^2 y^4 + 6\lambda x y^5 + y^6. \tag{6.1}$$

In the language of this paper, the four relevant results from [8, 16, 9] are these:

Proposition 6.1.
(i) $K_{2,4} = Q_{2,4}$.
(ii) $Q_{2,2r} \subsetneq K_{2,2r}$ *for* $r \ge 3$.
(iii) The elements of $\mathcal{E}(K_{2,6})$*, are* $[[q_\lambda]]$*, where* $0 < |\lambda| \le \frac{1}{2}$.
(iv) $K_{3,4} \subsetneq Q_{3,4}$*; specifically,* $x^4 + y^4 + z^4 + 6x^2y^2 + 6x^2z^2 + 2y^2z^2 \in K_{3,4} \setminus Q_{3,4}$.

According to [9], [8] gave a proof of (i) and (ii) (for even r); [9] gave a proof of (iii). All four appeared in [16]; (iii) was announced without proof. (The results from [16] were in the author's thesis, except that (iv) was proved there by an extremely long perturbation argument.) Note that (i) and (ii) follow from Prop. 3.8 and Theorems 3.10 and 3.11. Since $P_{n,m} = \Sigma_{n,m}$ if $n = 2$ or $(n,m) = (3,4)$, these examples are not helpful in resolving Parrilo's question about convex forms which are not sos.

The rest of this section discusses $\partial(K_{2,2r})$, mostly for small r. Let

$$p(x,y) = \sum_{i=0}^{2r} \binom{2r}{i} a_i x^{2r-i} y^i, \tag{6.2}$$

and define

$$\Theta_p(x,y) := \sum_{m=0}^{4r-4} b_m x^{4r-4-m} y^m, \quad \text{where}$$
$$b_m := \sum_{j=0}^{2r-1} \left(\binom{2r-2}{j}\binom{2r-2}{m-j} - \binom{2r-2}{j-1}\binom{2r-2}{m-j+1} \right) a_j a_{m+2-j}, \tag{6.3}$$

with the convention that $a_i = 0$ if $i < 0$ or $i > 2r$.

Proposition 6.2. [9, Prop.B] *Suppose* $p \in P_{2,2r}$*. Then* $p \in K_{2,2r}$ *if and only if* $\Theta_p \in P_{2,4r-2}$ *and* $p \in \partial(K_{2,2r})$ *if and only if* Θ_p *is psd but not pd.*

Proof. A direct computation shows that

$$\frac{\partial^2 p}{\partial x^2}\frac{\partial^2 p}{\partial y^2} - \left(\frac{\partial^2 p}{\partial x \partial y}\right)^2 = (2r)^2(2r-1)^2\Theta_p(x,y). \tag{6.4}$$

Since $Hes(p;u,u) = 2r(2r-1)p(u) \ge 0$, the first assertion is proved. Further, $p \in \partial(K_{2,2r})$ if and only if $Hes(p;u_0,v_0) = 0$ for some $u_0 \ne 0, v_0 \ne 0$. □

Observe that $\Theta_{(\alpha\cdot)^{2r}} = 0$, and it may be checked that if $q(x,y) = p(ax+by, cx+dy)$, then $\Theta_q(x,y) = (ad-bc)^2\Theta_p(ax+by, cx+dy)$. Thus, if $q \in \partial(K_{2,2r})$, we may assume that $q \sim p$, where $\Theta_p(0,1) = 0$, so that

$$0 = b_0 = a_0a_2 - a_1^2; \qquad 0 = b_1 = (2r-2)(a_0a_3 - a_1a_2). \tag{6.5}$$

We give a proof that $K_{2,4} = Q_{2,4}$, using the argument of [16] and, presumably, [8].

Proposition 6.3. $K_{2,4} = Q_{2,4}$.

Proof. Suppose $q \in \mathcal{E}(K_{2,4})$. Then $q \in \partial(K_{2,4})$ and $q \sim p$ where Θ_p is psd, but $\Theta_p(0,1) = 0$. If $a_0 = 0$, then $p(0,1) = 0$, so by Prop. 4.1, $p(x,y) = a_4 y^4$ is a 4th power. Otherwise, $a_0 > 0$, and if we write $a_1 = ra_0$, then by (6.5), we have $a_2 = r^2 a_0$ and $a_3 = r^3 a_0$. Write $a_4 = r^4 a_0 + s$. A computation shows that $\Theta_p(x,y) = a_0 s x^2 (x+ry)^2$, hence $s \geq 0$ and $p(x,y) = a_0(x+ry)^4 + sy^4$. Since $Q_{2,4} \subset K_{2,4}$ and $s \geq 0$, it follows that $p \in \mathcal{E}(K_{2,4})$ if and only if $s = 0$. Thus $p \in K_{2,4}$, being a sum of extremal elements, is a sum of 4th powers. □

If $2r = 6$, then we shall need $\Theta_p(x,y)$ in full bloom:

$$\begin{aligned}\Theta_p(x,y) = (a_0a_2 - a_1^2)x^8 + 4(a_0a_3 - a_1a_2)x^7y + (6a_0a_4 + 4a_1a_3 - 10a_2^2)x^6y^2 \\ +4(a_0a_5 + 4a_1a_4 - 5a_2a_3)x^5y^3 + (a_0a_6 + 14a_1a_5 + 5a_2a_4 - 20a_3^2)x^4y^4 \\ +4(a_1a_6 + 4a_2a_5 - 5a_3a_4)x^3y^5 + (6a_2a_6 + 4a_3a_5 - 10a_4^2)x^2y^6 \\ +4(a_3a_6 - a_4a_5)xy^7 + (a_4a_6 - a_5^2)y^8.\end{aligned} \tag{6.6}$$

Lemma 6.4. *If $p \in K_{2,6}$ and $\Theta_p(x,y) = \ell^2(x,y)B_p(x,y)$, where ℓ is linear and B_p is a pd sextic, then $p \notin \mathcal{E}(K_{2,6})$.*

Proof. After a linear change, we may assume $\ell(x,y) = y$, and assume p is given by (6.2), so that (6.6) holds. If $a_0 = p(1,0) = 0$, then as in Prop. 6.3, $p(x,y) = a_6 y^6$ and $\Theta_p(x,y) = 0$. Otherwise, we again have $a_1 = ra_0$, $a_2 = r^2 a_0$ and $a_3 = r^3 a_0$. A computation shows that

$$\begin{aligned}B_p(x,y) = 6a_0(a_4 - r^4a_0)x^6 + 4a_0(a_5 + 4ra_4 - 5r^5a_0)x^5y \\ +a_0(a_6 + 14ra_5 + 5r^2a_4 - 20r^6a_0)x^4y^2 \\ +4ra_0(a_6 + 4ra_5 - 5r^2a_4)x^3y^3 + (6r^2a_0a_4 + 4r^3a_0a_5 - 10a_4^2)x^2y^4 \\ +4(r^3a_0a_6 - a_4a_5)xy^5 + (a_4a_6 - a_5^2)y^6.\end{aligned} \tag{6.7}$$

Observe that if $p_\lambda = p + \lambda y^6$, then a_6 is replaced above by $a_6 + \lambda$ and

$$B_{p_\lambda} = B_p + \lambda(a_0x^4y^2 + 4ra_0x^3y^3 + 6r^2a_0x^2y^4 + 4r^3a_0xy^5 + a_4y^6). \tag{6.8}$$

Since B_p is pd, there exists sufficiently small ϵ so that $B_{p_{\pm\epsilon}}$ is psd, so $p_{\pm\epsilon} \in K_{2,6}$. But then $p = \frac{1}{2}(p_\epsilon + p_{-\epsilon})$ is not extremal. □

Proof of Prop. 6.1(iii). By Prop. 6.2 and Lemma 6.4, we may assume that $\Theta_p = y^2 B_p$ and B_p is psd, but not pd. If $B_p(0,1) = 0$, then by (6.7), $a_4 = r^4 a_0$ and $a_5 = r^5 a_0$ and, as before, if $a_6 = r^6 a_0 + t$, then $\Theta_p = atx^4(x+ry)^4$, so $t \geq 0$ and $p \in \mathcal{E}(K_{2,6})$ if and only if $t = 0$, so p is a 6th power.

If $B_p(1,e) = 0$ and $e \neq 0$, and $\tilde{p}(x,y) = p(y, x+ey)$, then $\Theta_{\tilde{p}}(x,y) = 0$ at $(x,y) = (1,0), (0,1)$, and by dropping the tilde, we may assume from (6.6) that $0 = a_4a_6 - a_5^2 = a_3a_6 - a_4a_5$. Again, $a_6 = p(0,1) \geq 0$, and if $a_6 = 0$, then p is a 6th power. Otherwise, we set $a_5 = sa_6$, so that $a_4 = s^2 a_6$ and $a_3 = s^3 a_6$; recall that $a_3 = r^3 a_0$ as well. If $s = 0$, then $a_3 = 0$, so $r = 0$ and $p(x,y) = a_0x^6 + a_6y^6$, which is only extremal if it is a 6th power. Thus $s \neq 0$, and similarly, $r \neq 0$. Letting $t = s^{-1}$, we obtain the formulation of [9]:

$$p(x,y) = a_0(x^6 + 6rx^5y + 15r^2x^4y^2 + 20r^3x^3y^3 + 15r^3tx^2y^4 + 6r^3t^2xy^5 + r^3t^3y^6) \tag{6.9}$$

Finally, send $(x,y) \mapsto (a_0^{-1/6}x, a_0^{-1/6}(rt)^{-1/2}y)$ and set $\lambda = \sqrt{r/t} = \sqrt{rs}$ to obtain q_λ.

A calculation shows that

$$\begin{gathered} \Theta_{q_\lambda}(x,y) = (1-\lambda^2)x^2y^2C_\lambda(x,y), \quad \text{where} \\ C_\lambda(x,y) = 6\lambda^2(x^4+y^4) + (4\lambda+20\lambda^3)(x^3y+xy^3) + (1+15\lambda^2+20\lambda^4)x^2y^2. \end{gathered} \tag{6.10}$$

Note that

$$\begin{gathered} D_\lambda(x,y) := C_\lambda(x+y,x-y) = (1+\lambda)(1+2\lambda)(1+5\lambda+10\lambda^2)x^4 \\ -2(1-\lambda^2)(1-20\lambda^2)x^2y^2 + (1-\lambda)(1-2\lambda)(1-5\lambda+10\lambda^2)x^4. \end{gathered} \tag{6.11}$$

If Θ_{q_λ} is psd, then $6\lambda^2(1-\lambda^2) \ge 0$, so $|\lambda| \le 1$. Under this assumption, it suffices to determine when D_λ is psd. Since $D_\lambda(1,0), D_\lambda(0,1) \ge 0$, $|\lambda| \le \frac{1}{2}$. If $D_\lambda(x,y) = E_\lambda(x^2,y^2)$, then the discriminant of E_λ is $128\lambda^2(1-\lambda^2)(1-10\lambda^2)$, hence D_λ is psd if $0 \le \lambda^2 \le \frac{1}{10}$. But, if $\frac{1}{20} \le \lambda^2 \le \frac{1}{4}$, then D_λ is a sum of psd monomials. Thus D_λ is psd if $|\lambda| \le \frac{1}{2}$, and hence this is also true for C_λ and thus for Θ_{q_λ}, so $q_\lambda \in K_{2,6}$. □

Since Θ_{q_λ} has two zeros when $|\lambda| < \frac{1}{2}$, but $\Theta_{q_{1/2}} = \frac{9}{8}x^2y^2(x+y)^2(x^2+xy+y^2)$ has three, one expects that the algebraic patterns for Θ_p will be variable for $p \in \mathcal{E}(K_{2,2r})$ for $r \ge 3$ and that $\mathcal{E}(K_{2,2r})$ will be hard to analyze.

Note also that

$$\begin{gathered} q_\lambda(x+y,x-y) = 2(1+\lambda)(1+5\lambda+10\lambda^2)x^6 + 30(1-\lambda^2)(1+2\lambda)x^4y^2 \\ +30(1-\lambda^2)(1-2\lambda)x^2y^4 + 2(1-\lambda)(1-5\lambda+10\lambda^2)y^6. \end{gathered} \tag{6.12}$$

One of the two boundary examples is $q_{-1/2}(x+y,x-y) = x^6 + 45x^2y^4 + 18y^6$, which scales to $x^6 + 15\alpha x^2y^4 + y^6$, where $\alpha^3 = \frac{1}{12}$.

We now consider the sections of $P_{2,6} = \Sigma_{2,6}$, $Q_{2,6}$ and $K_{2,6}$ consisting of forms

$$g_{A,B}(x,y) = x^6 + \binom{6}{2}Ax^4y^2 + \binom{6}{4}Bx^2y^4 + y^6, \tag{6.13}$$

and identify $g_{A,B}$ with the point (A,B) in the plane.

If $g_{A,B}$ is on the boundary of the $P_{2,6}$ section, then it is not pd, and we may assume $(x+ry)^2 \mid g_{A,B}$ for some $r \neq 0$. Thus, $(x-ry)^2 \mid g_{A,B}$ as well, and since the remaining factor must be even, the coefficients of x^6, y^6 force it to be $x^2 + \frac{1}{r^4}y^2$. Thus, the boundary forms for the section of $P_{2,6}$ are

$$(x^2 - r^2y^2)^2(x^2 + \tfrac{1}{r^4}y^2) = x^6 + (\tfrac{1}{r^4} - 2r^2)x^4y^2 + (r^4 - \tfrac{2}{r^2})x^2y^4 + y^6. \tag{6.14}$$

The parameterized boundary curve

$$(A,B) = \tfrac{1}{15}(\tfrac{1}{r^4} - 2r^2, r^4 - \tfrac{2}{r^2}) \tag{6.15}$$

is strictly decreasing as we move from left to right, and is a component of the curve $500(A^3+B^3) = 1875(AB)^2 + 150AB - 1$.

By (3.12), $g_{A,B}$ is in $Q_{2,6} = \Sigma^*_{2,6}$, iff $\begin{pmatrix} 1 & 0 & A & 0 \\ 0 & A & 0 & B \\ A & 0 & B & 0 \\ 0 & B & 0 & 1 \end{pmatrix}$ is psd iff $A \ge B^2$ and $B \ge A^2$, so the section is the familiar region between these two parabolas.

Except for the fortuitous identity (6.12), it would have been very challenging to determine the section for $K_{2,6}$. Scale x and y in (6.12) to get $g_{A,B}$: the parameterization of the boundary is $(\psi(\lambda), \psi(-\lambda))$, where

$$\psi(\lambda) = \frac{(1-\lambda)^{2/3}(1+\lambda)^{1/3}(1+2\lambda)}{(1+5\lambda+10\lambda^2)^{2/3}(1-5\lambda+10\lambda^2)^{1/3}}. \tag{6.16}$$

The intercepts occur when $\lambda = \pm\frac{1}{2}$ and are $(12^{-\frac{1}{3}}, 0)$ and $(0, 12^{-\frac{1}{3}})$. The point $(1,1)$ $(\lambda = 0)$ is smooth but of infinite curvature. The Taylor series of $\psi(\lambda)$ at $\lambda = 0$ begins $1 + \frac{16}{3}\lambda^3 - 48\lambda^4$, so locally, $x - y \approx \frac{32}{3}\lambda^3$ and $x + y - 2 \approx -96\lambda^4$, hence

$$x + y - 2 \approx -\tfrac{3^{7/3}}{2^{5/3}}(x-y)^{4/3}.$$

The maximum value of $\psi(\lambda)$ is $5^{-5/3}(1565 + 496\sqrt{10})^{1/3} \approx 1.000905$ at $\lambda = \frac{2\sqrt{10}-5}{15} \approx .0883$; this was asserted without proof in [16, p.232].

At this point, we punt and present some trinomials in $\partial(K_{2,2r})$. Suppose $1 \le v \le 2r-1$, $a, c > 0$ and suppose

$$h(x,y) = ax^{2r} + bx^{2r-v}y^v + cy^{2r} \in K_{2,2r}. \tag{6.17}$$

An examination of the end terms of Θ_h shows that v must be even and $b \ge 0$. If $b = 0$, then $h \in Q_{2,2r}$, so we assume $b > 0$, and wish to find the largest possible value of b. Calculations, which we omit, show that if

$$\begin{aligned} h_{r,k}(x,y) &:= (r-k)(2(r-k)-1)^2x^{2r} \\ &+ r(2r-1)(2k-1)(2r-2k-1)x^{2r-2k}y^{2k} + k(2k-1)^2y^{2r}, \end{aligned} \tag{6.18}$$

then $\Theta_{h_{r,k}}(x,y) = x^{2r-2-2k}y^{2k-2}(x^2-y^2)^2g(x,y)$, where g is a (psd) sum of even terms with positive coefficients, and that if $c > 0$ and $g_{r,k,c} = h_{r,k} + cx^{2r-2k}y^{2k}$, then $\Theta_{g_{r,k,c}}(1,1) < 0$. Given (a,c), there exist (α, β) so that the coefficients of x^{2r} and y^{2r} in $h_{r,k}(\alpha x, \beta y)$ are both 1, and we get the examples in [16, Prop.1]. In particular,

$$h_{4k,2k}(x,y) \sim x^{4k} + (8k-2)x^{2k}y^{2k} + y^{4k} \in \partial(K_{2,4k}). \tag{6.19}$$

Similar methods show that

$$x^{6k} + (6k-1)(6k-3)x^{4k}y^{2k} + (6k-1)(6k-3)x^{2k}y^{4k} + y^{6k} \in \partial(K_{2,6k}). \tag{6.20}$$

We have been unable to analyze $K_{2,8}$ completely, but have found this interesting element in $\mathcal{E}(K_{2,8})$:

$$p(x,y) = (x^2+y^2)^4 + \tfrac{8}{\sqrt{7}}\,xy(x^2-y^2)(x^2+y^2)^2, \tag{6.21}$$

for which $\Theta_p(x,y) = 3072x^2(x-y)^2y^2(x+y)^2(x^2+y^2)^2$.

7. Sums of 4th powers and octics

Hilbert's 17th Problem asks whether $p \in P_{n,2r}$ must be a sum of squares of rational functions: does there always exist $h = h_p \in F_{n,d}$ (for some d) so that $h^2 p \in \Sigma_{n,2r+2d} = W_{n,2(r+d)}$? Artin proved that the answer is "yes". (See [19, 21].) Becker [1] investigated the question for higher even powers. His result implies that if $p \in P_{2,2kr}$ and all real linear factors of p (if any) occur to an exponent which is a multiple of $2k$, then there exists $h = h_p \in F_{2,d}$ (for some d) so that $h^{2k}p \in W_{2,(r+d,2k)}$.

For example, by Becker's criteria, f_λ (c.f. (5.2)) is a sum of 4th powers of rational functions if and only if it is pd; that is, $\lambda \in (-\frac{1}{3}, \infty)$. As we have seen, $f_\lambda \in Q_{2,4} = W_{2,(1,4)}$ if and only if $\lambda \in [0,1]$. If ℓ is linear and $\ell^4 f = \sum_k h_k^4 \in W_{2,(2,4)}$, then $\ell | h_k$, so if $f_\lambda \notin Q_{2,4}$ and $h^4 f \in W_{2,(1+d,4)}$, then $\deg h = d \geq 2$. The identity

$$\begin{aligned} &3(3x^4 - 4x^2y^2 + 3y^4)(x^2+y^2)^4 \\ &= 2((x-y)^4 + (x+y)^4)(x^8+y^8) + 5x^{12} + 11x^8y^4 + 11x^4y^8 + 5y^{12} \end{aligned} \tag{7.1}$$

shows that $(x^2+y^2)^4 f_\lambda \in W_{2,(3,4)}$ for $\lambda \in [-\frac{2}{9}, \frac{11}{3}]$, since $T(-\frac{2}{9}) = \frac{11}{3}$, c.f. (5.4).

We know no alternate characterization of $W_{2,(u,4)}$, but offer the following conjecture:

Conjecture 7.1. *If $p \in P_{2,4u}$, then $p \in W_{2,(u,4)}$ if and only if there exist $f, g \in P_{2,2u}$ so that $p = f^2 + g^2$.*

It follows from (1.18) that the square of a psd binary form is a sum of three 4th powers. Conjecture 7.1 thus implies that any sum of 4th powers of polynomials is a sum of six 4th powers of polynomials. Any sum of s 4th powers will be a sum of s squares of psd forms; the conjecture asserts that p is a sum of *two* such squares. If $p \in W_{2,(u,4)}$, then $p \in P_{2,4u} = \Sigma_{2,4u}$, so $p = f^2 + g^2$ for some $f, g \in F_{n,2u}$; the conjecture says that there is a representation in which f and g are themselves psd.

This seems related to a result in [5] about sums of 4th powers of rational functions over real closed fields. If $p = \sum h_k^4$ and $\ell | p$ for a linear form, then $\ell^{4t} | p$ for some t and $\ell^t | h_k$, so we may assume p is pd. The following is a special case of [5, Thm.4.12], referring to sums of 4th powers of non-homogeneous rational functions.

Proposition 7.2. *Suppose $p \in \mathbf{R}[x]$ is pd. Then p is a sum of 4th powers in $\mathbf{R}(x)$ if and only if there exist pd f, g, h in $\mathbf{R}[x]$, $\deg f = \deg g$, such that $h^2 p = f^2 + g^2$.*

It follows that a sum of 4th powers in $\mathbf{R}(x)$ is a sum of at most six 4th powers.

Theorem 7.3. *Conjecture 7.1 is true for $p \in W_{2,(1,4)} = Q_{2,4}$.*

Proof. We have seen that if $p \in W_{2,(1,4)}$, then $p \sim f_\lambda$ for $\lambda \in [0,1]$. If $\lambda \in (\frac{1}{3}, 1]$, then $T(\lambda) \in [0, \frac{1}{3})$, so it suffices to find a representation for F_λ with $\lambda \in [0, \frac{1}{3}]$. Such a representation is $f_\lambda(x,y) = (x^2 + 3\lambda y^2)^2 + (1 - 9\lambda^2)(y^2)^2$. □

Theorem 7.4. *Conjecture 7.1 is true for even symmetric octics.*

It will take some work to get to the proof of Theorem 7.4. For the rest of this section, write $W := W_{2,(2,4)}$. We first characterize $\partial(W^*)$.

Theorem 7.5. *If $p \in \partial(W^*)$, then $p = (\alpha\cdot)^8$ or $p \sim q$, where*

$$q(x,y) = d_0x^8 + 8d_1x^7y + 28d_2x^6y^2 + 28d_6x^2y^6 + 8d_7xy^7 + d_8y^8, \tag{7.2}$$

and

$$(6d_2u^2 + 6d_6w^2)(d_0u^4 + 4d_2u^3w + 4d_6uw^3 + d_8w^4) - (2d_1u^3 + 2d_7w^3)^2 \tag{7.3}$$

is psd.

Proof. Consider a typical element $q \in W^*$,

$$q(x,y) = \sum_{k=0}^{8} \binom{8}{k} d_k x^{8-k} y^k. \tag{7.4}$$

Then as in Prop. 3.9,

$$\begin{aligned} H_q(u,v,w) := [q,(ux^2+vxy+wy^2)^4] &= d_0u^4 + 4d_1u^3v + d_2(6u^2v^2 + 4u^3w) \\ +d_3(4uv^3 + 12u^2vw) + d_4(v^4 &+ 12uv^2w + 6u^2w^2) + d_5(4v^3w + 12uvw^2) \\ &+d_6(6v^2w^2 + 4uw^3) + 4d_7vw^3 + d_8w^4 \end{aligned} \tag{7.5}$$

is a psd ternary quartic in u,v,w. If $q \in \partial(W^*)$, then $[q,h^2] = 0$ for some non-zero quadratic h. Since $\pm h \sim x^2, xy, x^2+y^2$, it suffices by Prop. 2.6 to consider three cases: $[q,x^8] = 0, [q,x^4y^4] = 0$ and $[q,(x^2+y^2)^4] = 0$. Since

$$420(x^2+y^2)^4 = 256(x^8+y^8) + \sum_{\pm}(x \pm \sqrt{3}y)^8 + (\sqrt{3}x \pm y)^8, \tag{7.6}$$

$[q,(x^2+y^2)^4] = 0$ implies that $q(1,0) = q(0,1) = q(1,\pm\sqrt{3}) = q(\sqrt{3},\pm 1) = 0$; since q is psd, $q = 0$. (An alternate proof derives this result from $(x^2+y^2)^4 \in int(Q_{2,8})$ by [18, Thm.8,15(ii)], so $(x^2+y^2)^4 \in int(W)$.)

Suppose $[h,(x^2)^4] = 0$; that is, $H_q(1,0,0) = 0$. Then $d_0 = 0$, and since H_q is now at most quadratic in u, it follows that $d_1 = d_2 = 0$. This implies that the coefficient of u^2 in H_q is $12d_3vw + 6d_4w^2$, hence $d_3 = 0$ and

$$\begin{aligned} H_q(u,v,w) &= u^2(6d_4w^2) + 2u(2d_6w^3 + 6d_5vw^2 + 6d_4v^2w) \\ &+(d_8w^4 + 4d_7w^3v + 6d_6w^2v^2 + 4d_5wv^3 + d_4v^4). \end{aligned} \tag{7.7}$$

Since H_q is psd if and only if its discriminant with respect to u is psd in v,w, and this discriminant is $-30d_4^2v^4w^2+$ lower terms in v, $d_4 = 0$. Since H_q cannot be linear in u, it follows that $d_5 = d_6 = 0$ and $H_q(u,v,w) = d_8w^4 + 4d_7w^3v$, which is only psd if $d_7 = 0$, so that $q(x,y) = d_8y^8$ is an 8th power.

Finally, suppose $[q,x^4y^4] = 0$; that is, $H_q(0,1,0) = d_4 = 0$. Since H_q is at most quadratic in v, it follows that $d_3 = d_5 = 0$ as well, so q has the shape (7.2) and

$$\begin{aligned} H_q(u,v,w) &= v^2(6d_2u^2 + 6d_6w^2) \\ +2v(2d_1u^3 + 2d_7w^3) &+ d_0u^4 + 4u^3wd_2 + 4uw^3d_6 + d_8w^4; \end{aligned} \tag{7.8}$$

H_q is psd if and only if its discriminant with respect to v, namely (7.3), is psd. $\square$

It should be possible to characterize $\mathcal{E}(W^*)$, though we do not do so here. One family of extremal elements is parameterized by $\alpha \in \mathbf{R}$:

$$\omega_\alpha(x,y) := x^8 + 28x^2y^6 + 24\alpha xy^7 + 3(1+2\alpha^2)y^8 \in \mathcal{E}(W^*). \tag{7.9}$$

In this case,

$$\begin{aligned} H_{\omega_\alpha}(u,v,w) &= 6v^2w^2 + 12\alpha vw^3 + u^4 + 4uw^3 + (3+6\alpha^2)w^4 \\ &= 6(vw+\alpha w^2)^2 + (u+w)^2(u^2-2uw+3w^2) \end{aligned} \tag{7.10}$$

is psd; $H_{\omega_\alpha}(0,1,0) = H_{\omega_\alpha}(1,\alpha,-1) = 0$, and $H_{\omega_\alpha}(u,v,0) = u^4$ has a 4th order zero at $(0,1,0)$. It is unclear whether ω_α has other interesting algebraic properties.

We now simplify matters by limiting our attention to even symmetric octics. Let

$$\widetilde{F} = \{((A,B,C)) := Ax^8 + Bx^6y^2 + Cx^4y^4 + Bx^2y^6 + Ay^8 \ : \ A,B,C \in \mathbf{R}\}. \tag{7.11}$$

denote the cone of even symmetric octics, and let

$$\widetilde{W} = W \cap \widetilde{F}. \tag{7.12}$$

Then $\widetilde{W}$ is no longer a blender, because (P3) fails spectacularly. However, it is still a closed convex cone. We give the inner product explicitly:

$$p_i = ((A_i,B_i,C_i)) \implies [p_1,p_2] = A_1A_2 + \tfrac{B_1B_2}{28} + \tfrac{C_1C_2}{70} + \tfrac{B_1B_2}{28} + A_1A_2. \tag{7.13}$$

Let $(\widetilde{W})^* \subset \widetilde{F}$ denote the dual cone to $\widetilde{W}$. Here is a special case of [18, p.142].

Theorem 7.6. *$(\widetilde{W})^* = W^* \cap \widetilde{F}$.*

Proof. Suppose $p \in \widetilde{W}$ and $q \in W^* \cap \widetilde{F}$. Then $p \in W$ and $q \in W^*$ imply $[p,q] \ge 0$, so $q \in (\widetilde{W})^*$. Suppose now that $q \in (\widetilde{W})^*$; we wish to show that $q \in W^*$. Pick $r \in W$, and let $r_1 = r$, $r_2(x,y) = r(x,-y)$, $r_3(x,y) = r(y,x)$ and $r_4(x,y) = r(y,-x)$. Since $q \in \widetilde{F}$, $[r_j,q] = [r,q]$ for $1 \le j \le 4$, and since $p = r_1+r_2+r_3+r_4 \in \widetilde{W}$, $0 \le [p,q] = 4[r,q]$. Thus, $[r,q] \ge 0$ as desired. □

We need not completely analyze $(\widetilde{W})^*$ to determine $\widetilde{W}$. The following suffices.

Lemma 7.7. *If $q = ((1,0,0))$, $((4,28,0))$ or $((6-4\lambda^2+3\lambda^4, 28(6-\lambda^2), 420))$, $\lambda \in \mathbf{R}$, then $q \in W^*$.*

Proof. Using the notation of (7.4), suppose

$$q(x,y) = ((d_0, 28d_2, 70d_4)) = d_0^8 + 28d_2x^6y^2 + 70d_4x^4y^4 + 28d_2x^2y^6 + d_0y^8. \tag{7.14}$$

Comparison with (7.13) shows that

$$q \in \widetilde{W}^* \iff ((A,B,C)) \in \widetilde{W} \implies 2d_0A + 2d_2B + d_4C \ge 0. \tag{7.15}$$

On the other hand, (7.5) and Theorem 7.6 imply that $q \in \widetilde{W}^*$ if and only if

$$H_q(u,v,w) = d_0(u^4+w^4) + d_2(u^2+w^2)(6v^2+4uw) + d_4(v^4+12uv^2w+6u^2w^2) \tag{7.16}$$

is psd. If $(d_0, d_2, d_4) = (1, 0, 0)$, then $H_q(u, v, w) = u^4 + w^4$, which is psd, and if $(d_0, d_2, d_4) = (4, 1, 0)$, then

$$H_q(u, v, w) = 4(u + w)^2(u^2 - uw + w^2) + 6(u^2 + w^2)v^2. \tag{7.17}$$

Finally, if $(d_0, d_2, d_4) = (6 - 4\lambda^2 + 3\lambda^4, 6 - \lambda^2, 6)$, then a computation gives

$$\begin{aligned} 2H_q(u, v, w) &= 2(6 - 4\lambda^2 + 3\lambda^4)(u^4 + w^4) \\ &+2(6 - \lambda^2)(u^2 + w^2)(6v^2 + 4uw) + 12(v^4 + 12uv^2w + 6u^2w^2) \\ &= 48(u + w)^2v^2 + 4\lambda^2(u + w)^4 + 3\lambda^4(u^2 - w^2)^2 \\ &+3(2v^2 + 2(u + w)^2 - \lambda^2(u^2 + w^2))^2. \end{aligned} \tag{7.18}$$

Note that $H_q(1, \pm\lambda, -1) = 0$. □

An important family of elements in $\widetilde{W}$ is

$$\begin{aligned} \psi_\lambda(x, y) &:= \tfrac{1}{2}\left((x^2 + \lambda xy - y^2)^4 + (x^2 - \lambda xy - y^2)^4\right) \\ &= ((1,\ 6\lambda^2 - 4,\ \lambda^4 - 12\lambda^2 + 6)) \end{aligned} \tag{7.19}$$

Theorem 7.8. *The extremal elements of $\widetilde{W}$ are x^4y^4 and $\{\psi_\lambda : \lambda \ge 0\}$. Hence $p = ((A, B, C)) \in \widetilde{W}$ if and only if*

$$A = B = 0,\ C \ge 0, \quad \text{or} \quad A > 0,\ B \ge -4A,\ 36AC \ge B^2 - 64AB - 56A^2. \tag{7.20}$$

Proof. By Lemma 7.7 and (7.15), if $p \in \widetilde{W}$, then $A \ge 0$, $A + 4B \ge 0$ and

$$2(6 - 4\lambda^2 + 3\lambda^4)A + 2(6 - \lambda^2)B + 6C \ge 0. \tag{7.21}$$

We have $A = p(1, 0) = p(0, 1) \ge 0$, and if $A = 0$ and $p = \sum h_k^4$, then $xy|h_k$, hence $p = [0, 0, C]$ with $C \ge 0$. Otherwise, assume that $A = 1$, so that (7.20) becomes

$$B \ge -4, \quad C \ge \tfrac{1}{36}(B^2 - 64B - 56). \tag{7.22}$$

The first inequality follows from $((4, 28, 0)) \in \widetilde{W}^*$, and we can thus write $B = 6\alpha^2 - 4$, where $\alpha = \sqrt{\frac{B+4}{6}}$. Put $\lambda = \alpha$ in (7.21) to obtain

$$C \ge \alpha^4 - 12\alpha^2 + 6 = \tfrac{1}{36}(B^2 - 64B - 56). \tag{7.23}$$

Conversely, suppose $p = ((A, B, C))$ satisfies (7.20). If $A = 0$, then $p = cx^4y^4 \in \widetilde{W}$. If $A > 0$, then we can take $A = 1$ and substitute $B = 6\alpha^2 - 4$, so that, by (7.23),

$$p = ((1, B, C)) = ((1, 6\alpha^2 - 4, \alpha^4 - 12\alpha^2 - 6)) + ((0, 0, \gamma)) = \psi_\lambda(x, y) + \gamma x^4y^4 \tag{7.24}$$

for some $\gamma \ge 0$, hence $p \in \widetilde{W}$. □

Taking $(A, B) = (1, 0)$, we obtain (1.19). Suppose $\lambda, \mu \ge -2$. Then Theorem 7.6 implies that (c.f. (5.2)) $f_\lambda(x, y)f_\mu(x, y) \in W$ if and only if

$$(17 - 12\sqrt{2})(\lambda + 2) \le \mu + 2 \le (17 + 12\sqrt{2})(\lambda + 2) \tag{7.25}$$

There is a peculiar resonance with the example after Theorem 4.7.

Proof of Theorem 7.4. Suppose the even symmetric octic $((A,B,C))$ satisfies (7.20). If $A=0$, then $((0,0,C)) = C(x^2y^2)^2$. Otherwise, again suppose $A=1$ and write $B = 6\alpha^2 - 4$, so

$$B = 6\alpha^2 - 4, \quad C = \tfrac{1}{36}(B^2 - 64B - 56) + T = \alpha^4 - 12\alpha^2 + 6 + T, \quad T \ge 0. \tag{7.26}$$

Observe that

$$\begin{aligned}&(x^4 + (3\alpha^2-2)x^2y^2 + y^4)^2 + (T - 8\alpha^4)(x^2y^2)^2 \\ &= ((1, 6\alpha^2 - 4, 9\alpha^4 - 12\alpha^2 + 6)) + ((0,0,T-8\alpha^4)) = ((1,B,C)),\end{aligned} \tag{7.27}$$

so if $T \ge 8\alpha^4$, then we are done. Otherwise, $0 \le T \le 8\alpha^4$. Finally, note that

$$\begin{aligned}&\tfrac{1}{2}\left(\left((x^2 - \sqrt{\lambda}xy - y^2)^2 + \mu x^2y^2\right)^2 + \left((x^2 + \sqrt{\lambda}xy - y^2)^2 + \mu x^2y^2\right)^2\right) \\ &= ((1, 6\lambda + 2\mu - 4, 6 - 12\lambda + \lambda^2 - 4\mu + 2\lambda\mu + \mu^2))\end{aligned} \tag{7.28}$$

is a sum of two squares of psd forms if $\mu \ge 0$. One solution to the system

$$6\alpha^2 - 4 = 6\lambda + 2\mu - 4, \alpha^4 - 12\alpha^2 + 6 + T = 6 - 12\lambda + \lambda^2 - 4\mu + 2\lambda\mu + \mu^2 \tag{7.29}$$

is

$$\lambda = \frac{3\alpha^2 - \sqrt{\alpha^4 + T}}{2}, \quad \mu = \frac{3(\sqrt{\alpha^4+T} - \alpha^2)}{2}. \tag{7.30}$$

Evidently, $\mu \ge 0$; since $T \le 8\alpha^4$, $\lambda \ge 0$, so $\sqrt{\lambda}$ is real. □

8. Bibliography

References

[1] E. Becker, *The real holomorphy ring and sums of 2nth powers*, Real algebraic geometry and quadratic forms (Rennes, 1981), pp. 139–181, Lecture Notes in Math., **959**, Springer, Berlin-New York, 1982, MR0683132 (84g:12032).

[2] G. Blekherman, *Convex forms that are not sums of squares*, arXiv: 0910.065v1.

[3] M. D. Choi and T. Y. Lam, *An old question of Hilbert*, Queen's Papers in Pure and Appl. Math. (Proceedings of Quadratic Forms Conference, Queen's University (G. Orzech ed.)), **46** (1976), 385–405, MR0498375 (58 #16503).

[4] M. D. Choi and T. Y. Lam, *Extremal positive semidefinite forms*, Math. Ann., **231** (1977/78), 1–18, MR0498384 (58 #16512).

[5] M. D. Choi, T. Y. Lam, A. Prestel and B. Reznick, *Sums of 2m-th powers of rational functions in one variable over real closed fields*, Math. Z., **221** (1996), 93-112, MR1369464 (96k:12003).

[6] M. D. Choi, T. Y. Lam and B. Reznick, *Positive sextics and Schur's inequalities*, J. Algebra, **141** (1991), 36–77, MR1118315 (92i:11045).

[7] M. D. Choi, T. Y. Lam and B. Reznick, *Sums of squares of real polynomials*, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), 103–126, Proc. Sympos. Pure Math., **58**, Part 2, Amer. Math. Soc., Providence, RI, 1995, MR1327293 (96f:11058).

[8] V. I. Dmitriev, *The structure of a cone in a five-dimensional space*, (Russian) Vorone. Gos. Univ. Trudy Naun.-Issled. Inst. Mat. VGU Vyp. 7 (1973), 13–22, MR0467523 (57 #7379).

[9] V. I. Dmitriev, *Extreme rays of a cone of convex forms of the sixth degree in two variables*, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1991, no. 10, 28–35; translation in Soviet Math. (Iz. VUZ) 35 (1991), no. 10, 25–31, MR1179211 (93i:12003).

[10] W. J. Ellison, *A 'Waring's problem' for homogeneous forms*, Proc. Cambridge Philos. Soc., **65** (1969), 663-672, MR0237450 (38 #5732).

[11] W. Harris, *Real even symmetric ternary forms*, J. Algebra, **222** (1999), 204–245, MR1728161 (2001a:11070).

[12] D. Hilbert, *Über die Darstellung definiter Formen als Summe von Formenquadraten*, Math. Ann., **32** (1888), 342–350; see Ges. Abh. 2, 154–161, Springer, Berlin, 1933, reprinted by Chelsea, New York, 1981.

[13] E. R. Lorch, *Differentiable inequalities and the theory of convex bodies*, Trans. Amer. Math. Soc., **71**, (1951), 243–266, MR0052804 (14,678g).

[14] V. Powers and B. Reznick, *Notes towards a constructive proof of Hilbert's theorem on ternary quartics*, Quadratic forms and their applications (Dublin, 1999), 209–227, Contemp. Math., **272**, Amer. Math. Soc., Providence, RI, 2000, MR1803369 (2001h:11049).

[15] B. Reznick, *Banach spaces which satisfy linear identities*, Pacific J. Math., **74** (1978), 221–233, MR0487407 (58 #7045).

[16] B. Reznick, *Banach spaces with polynomial norms*, Pacific J. Math., **82** (1979), 223–235, MR0549846 (83c:46007).

[17] B. Reznick, *Forms derived from the arithmetic-geometric inequality*, Math. Ann., **283** (1989), 431–464, MR0985241 (90i:11043).

[18] B. Reznick, *Sums of even powers of real linear forms*, Mem. Amer. Math. Soc., **96** (1992), no. 463, MR1096187 (93h:11043).

[19] B. Reznick, *Uniform denominators in Hilbert's Seventeenth Problem*, Math. Z., **220** (1995), 75-97, MR1347159 (96e:11056).

[20] B. Reznick, *Homogeneous polynomial solutions to constant coefficient PDE's*, Adv. Math., **117** (1996), 179–192, MR1371648 (97a:12006).

[21] B. Reznick, *Some concrete aspects of Hilbert's 17th Problem*, Real algebraic geometry and ordered structures (Baton Rouge, LA, 1996), 251–272, Contemp. Math., **253**, Amer. Math. Soc., Providence, RI, 2000, MR1747589 (2001i:11042).

[22] B. Reznick, *On Hilbert's construction of positive polynomials*, arXiv:0707.2156.

[23] R. T. Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28 Princeton University Press, Princeton, N.J. 1970, MR0274683 (43 #445).

[24] R. Sanyal, F. Sottile and B. Sturmfels, *Orbitopes*, arXiv:0911.5436v3.

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