# ON THE ABSENCE OF UNIFORM DENOMINATORS IN HILBERT'S 17TH PROBLEM

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ABSTRACT. Hilbert showed that for most $(n,m)$ there exist psd forms $p(x_1,\dots,x_n)$ of degree $m$ which cannot be written as a sum of squares of forms. His 17th problem asked whether, in this case, there exists a form $h$ so that $h^2p$ is a sum of squares of forms; that is, $p$ is a sum of squares of rational functions with denominator $h$. We show that, for every such $(n,m)$ there does not exist a single form $h$ which serves in this way as a denominator for *every* psd $p(x_1,\dots,x_n)$ of degree $m$.

## 1. INTRODUCTION

Let $H_d(\mathbb{R}^n)$ denote the set of real homogeneous forms of degree $d$ in $n$ variables ("$n$-ary $d$-ics") . By identifying $p \in H_d(\mathbb{R}^n)$ with the $N = \binom{n+d-1}{n-1}$-tuple of its coefficients, we see that $H_d(\mathbb{R}^n) \approx \mathbb{R}^N$. Suppose $m$ is an even integer. A form $p \in H_m(\mathbb{R}^n)$ is called *positive semidefinite* or *psd* if $p(x_1,\dots,x_n) \ge 0$ for all $(x_1,\dots,x_n) \in \mathbb{R}^n$. Following [1], we denote the set of psd forms in $H_m(\mathbb{R}^n)$ by $P_{n,m}$. Since $P_{n,m}$ is closed under addition and closed under multiplication by positive scalars, it is a convex cone. In fact, $P_{n,m}$ is a *closed* convex cone: if $p_n \to p$ coefficient-wise, and each $p_n$ is psd, then so is $p$. A psd form is called *positive definite* or *pd* if $p(x_1,\dots,x_n) = 0$ implies $x_j = 0$ for $1 \le j \le n$. The pd $n$-ary $m$-ics are the interior of the cone $P_{n,m}$.

A form $p \in H_m(\mathbb{R}^n)$ is called *a sum of squares* or *sos* if it can be written as a sum of squares of polynomials; that is, $p = \sum_k h_k^2$. It is easy to show in this case that each $h_k \in H_{m/2}(\mathbb{R}^n)$. Again following [1], we denote the set of sos forms in $H_m(\mathbb{R}^n)$ by $\Sigma_{n,m}$. Clearly, $\Sigma_{n,m}$ is a convex cone; less obviously, it is a closed cone, a result due to R. M. Robinson [20].

In light of the inclusion $\Sigma_{n,m} \subseteq P_{n,m}$, let $\Delta_{n,m} = P_{n,m} \setminus \Sigma_{n,m}$. It was well-known by the late 19th century that $P_{n,m} = \Sigma_{n,m}$ when $m = 2$ or $n = 2$. In 1888, Hilbert proved [8] that $\Sigma_{3,4} = P_{3,4}$; more specifically, every $p \in P_{3,4}$ can be written as the sum of three squares of quadratic forms. (An elementary proof, with "five" squares is in [2, pp.16-17]; for modern expositions of Hilbert's proof, see [24] and [21].) Hilbert also proved in [8] that the preceding are the *only* cases for which $\Delta_{n,m} = \emptyset$. That is,

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if $n \ge 3$ and $m \ge 6$ or $n \ge 4$ and $m \ge 4$, then there exist psd forms $n$-ary $m$-ics that are not sos.

In 1893, Hilbert [9] generalized his three-square result for $P_{3,4}$ to ternary forms of higher degree. Suppose $p \in P_{3,m}$ with $m \ge 6$. Then there exist $p_1 \in P_{3,m-4}$ and $h_{1k} \in H_{m-2}(\mathbb{R}^3)$, $1 \le k \le 3$, so that

$$p_1 p = h_{11}^2 + h_{12}^2 + h_{13}^2.$$

(Hilbert's proof seems to be non-constructive, and lacks a modern exposition. In the very recent paper [10], de Klerk and Pasechnik discuss the implementation of an algorithm to find $p_1$ so that $p_1 p$ is sos, though not necessarily as a sum of *three* squares. This paper uses Hilbert's result without giving an independent proof.)

If $m = 6$ or 8, then $p_1$ is a sum of three squares of forms, and hence (as Landau later noted [11]), the four-square identity implies that $p_1^2 p = p_1(p_1 p)$ is the sum of four squares of forms. If $m \ge 10$, then the argument can be applied to $p_1$: there exists $p_2 \in P_{3,m-8}$ with $p_2 p_1 = h_{21}^2 + h_{22}^2 + h_{23}^2$. Thus, if $m = 10$ or 12 (so that $P_{3,m-8} = \Sigma_{3,m-8}$), then $(p_1 p_2)^2 p = p_2(p_2 p_1)(p_1 p)$ is the sum of four squares of forms, An easy induction shows that there exists $q \in H_t(\mathbb{R}^3)$ with $t = \lfloor \frac{(m-2)^2}{8} \rfloor$ so that $q^2 p$ is the sum of four squares of forms.

Hilbert's 17th Problem asked whether this generalizes to $n > 3$ variables; that is, if $p \in P_{n,m}$, must there exist some form $q$ so that $q^2 p$ is sos? Artin proved that there must be, in a way that gives no information about $q$. Much more on the history of this subject can be found in the survey paper [19].

This discussion leads to two closely related questions. Suppose $p \in P_{n,m}$. Can we *find* a form $h$ such that $hp$ is sos? Can we *find* a form $q$ so that $q^2 p$ is sos? If we've answered the second, we've answered the first. Conversely, if $p \ne 0$ is psd and $hp$ is sos, then $h$ is psd. But it needn't be sos; indeed, a trivial answer to the first question is to take $h = p$. Stengle proved [23] that if $p(x,y,z) = x^3 z^3 + (y^2 z - x^3 - z^2 x)^2$, then $p^{2s+1} \in \Delta_{3,6(2s+1)}$ for every integer $s$. That is, $p^{2s-1} \cdot p$ is sos, but $p^{2s} \cdot p$ is not. Choi and Lam showed [1] that for $S \in \Delta_{3,6}$ (see (3) below), the product $S(x,y,z)S(x,z,y)$ is actually sos.

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## 2. What is known about the denominator

The first concrete result about a denominator in Hilbert's 17th Problem was found by Pólya [16]. He showed that if $f \in H_d(\mathbb{R}^n)$ is positive on the unit simplex $\{(x_1, \dots, x_n) \mid x_j \ge 0, \sum x_j = 1\}$, then for sufficiently large $N$, $(\sum_j x_j)^N f$ has positive coefficients. Replacing each $x_j$ by $x_j^2$, we see that if $p \in H_{2d}(\mathbb{R}^n)$ is an even positive definite form, then $(\sum_j x_j^2)^N p$ is a sum of even monomials with positive coefficients, and so, as it stands, is a sum of squares of monomials. Taking even $N$, we see that $q = (\sum_j x_j^2)^{N/2}$ is a denominator for $p$. Habicht [6] generalized Pólya's proof to give an alternate solution to Hilbert's 17th Problem for pd forms; however, $h$ is not readily constructible and in general is no longer a power of $\sum x_j^2$. Except for one example, Pólya did not attempt to determine an explicit value of $N$. A good exposition of the theorems of Pólya and Habicht can be found in [7].

For positive definite $p \in P_{n,m}$, let

$$\epsilon(p) := \frac{\inf\{p(u) : u \in S^{n-1}\}}{\sup\{p(u) : u \in S^{n-1}\}}$$

measure how "close" $p$ is to having a zero. The author [18] showed that if

$$N \geq \frac{nm(m-1)}{(4\log 2)\epsilon(p)} - \frac{n+m}{2},$$

then $(\sum_j x_j^2)^N p$ is a sum of $(m+2N)$-th powers of linear forms, and so is sos. A similar lower bound has been shown to apply in Pólya's case, one which goes to infinity as $p$ approaches the boundary of $P_{n,m}$. (See papers by de Loera and Santos [12] and by Powers and the author [17].)

The restriction to positive definite forms is necessary. There exist psd forms $p$ in $n \geq 4$ variables so that, if $h^2 p$ is sos, then $h$ must have a specified zero. The existence of these unavoidable singularities, or so-called "bad points", insures that $(\sum x_j^2)^r p$ can never be a sum of squares of forms for *any* $r$. Habicht's Theorem implies that no positive definite form can have a bad point. Bad points were first noted by Straus and have been extensively studied by Delzell; see, e.g. [4, 5].

## 3. Recent results and a new theorem

Scheiderer has shown in very recent work [22] that for $p \in P_{3,m}$, there exists $N = N(p)$ so that $(x^2+y^2+z^2)^N p(x,y,z)$ is sos; indeed, $x^2+y^2+z^2$ can be replaced by any positive definite form. This is a strong refutation to the existence of bad points for ternary forms.

Also very recently, Lombardi and Roy [13] have constructed a quantitative version of the Positivstellensatz. A special case is that for fixed $(n,m)$, there exists $d = d(n,m)$ so that if $p \in P_{n,m}$, there exists $q \in H_d(\mathbb{R}^n)$ so that $q^2 p$ is sos.

Suppose $(n,m)$ is such that $\Delta_{n,m} \neq \emptyset$. Theorem 1 below states that there is no *single* form $h$ so that, if $p \in P_{n,m}$, then $hp$ is sos. Corollary 2 says that there is not even a *finite* set of forms $\mathcal{H}$ so that, if $p \in P_{n,m}$, then there exists $h \in \mathcal{H}$ so that $hp$ in sos. In particular, there does not exist a finite set of denominators which apply to all of $P_{n,m}$. This result implies that $N(p)$ in Scheiderer's theorem is not bounded as $p$ ranges over $P_{3,m}$. It also implies that the denominators in the Lombardi-Roy theorem cannot be chosen from a finite, predetermined set.

The proof of the Theorem is elementary and relies on a few simple observations. If $p \neq 0$ is psd and $hp$ is sos, then $h$ is psd. As previously noted, $\Sigma_{n,m}$ is a closed cone for all $(n,m)$. This cone is invariant under the action of taking invertible linear changes of form. Thus, if $h'$ is derived from $h$ by such a linear change, and if $hp$ is sos for every $p \in P_{n,m}$, then so is $h'p$. Suppose $\ell$ is a linear form, $p = \sum_j g_k^2$ is sos, and $\ell \mid p$. Then $\ell^2 \mid p$ and $\ell \mid g_k$ for each $k$, and by induction, $\ell^{2s} \mid p \implies \ell^s \mid g_k$. Thus, we can "peel off" squares of linear factors from any sos form; this is a common practice, dating back at least to [20, p. 267]. We use this observation in the contrapositive: if $p \in \Delta_{n,m}$, then $\ell^{2s}p \in \Delta_{n,m+2s}$.

**Theorem 1.** *Suppose $\Delta_{n,m} \neq \emptyset$. Then there does not exist a non-zero form $h$ so that if $p \in P_{n,m}$, then $hp$ is sos.*

*Proof.* Suppose to the contrary that such a form $h$ exists. Since $h \neq 0$, there exists a point $a \in \mathbb{R}^n$ so that $h(a) \neq 0$. By making an invertible linear change of variables, we can take $a = (1, 0, \dots, 0)$. Thus, we may assume without loss of generality that $h(x_1, 0, \dots, 0) = \alpha x_1^d$, where $\alpha > 0$ and $d$ is even. In the sequel, we distinguish $x_1$ from the other variables.

Choose $p \in P_{n,m} \setminus \Sigma_{n,m}$. Then

$$h(x_1, x_2, \dots, x_n)p(x_1, rx_2, \dots, rx_n)$$

is sos for every $r \in \mathbb{N}$. By making the change of variables $x_i \to x_i/r$ for $i \geq 2$, we see that

$$h(x_1, r^{-1}x_2, \dots, r^{-1}x_n)p(x_1, x_2, \dots, x_n)$$

is also sos. Since

$$\lim_{r\to\infty} h(x_1, r^{-1}x_2, \dots, r^{-1}x_n) = h(x_1, 0, \dots, 0) = \alpha x_1^d,$$

and since $\Sigma_{n,m+d}$ is closed, it follows that

$$\lim_{r\to\infty} h(x_1, r^{-1}x_2, \dots, r^{-1}x_n)p(x_1, x_2, \dots, x_n) = \alpha x_1^d p(x_1, \dots, x_n)$$

is sos. Thus $p$ is sos, a contradiction. □

The following elegant proof is due to Claus Scheiderer and is included with his permission; it supersedes the proof in an earlier version of this manuscript.

**Corollary 2.** *Suppose $\Delta_{n,m} \neq \emptyset$. Then there does not exist a finite set of non-zero forms $\mathcal{H} = \{h_1, ..., h_N\}$ with the property that, if $p \in P_{n,m}$, then $h_k p$ is sos for some $h_k \in \mathcal{H}$.*

*Proof.* Suppose $\mathcal{H}$ exists. For each $k$, there exists non-zero $p \in \Delta_{n,m}$ so that $h_k p$ is sos. (Otherwise, we may delete $h_k$ harmlessly from $\mathcal{H}$.) Thus, each $h_k$ is psd, and there exists a form $q_k$ so that $q_k^2 h_k$ is sos. Define $h = \prod_k q_k^2 h_k$. We now show that for every $p \in P_{n,m}$, $hp$ is sos: this contradicts the Theorem and proves the Corollary. By hypothesis, there exists $h_j \in \mathcal{H}$ so that $h_j p$ is sos. Thus,

$$hp = \left(\prod_{k\neq j} q_k^2 h_k\right) \cdot q_j^2 \cdot h_j p$$

is a product of sos factors, and so is sos. □

Finally, we know by Hilbert's theorem that for $p \in P_{3,6}$, there exists quadratic $h$ so that $hp \in \Sigma_{3,8}$. The three simplest forms in $\Delta_{3,6}$ are

$$M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2, \quad \text{due to Motzkin [14];} \tag{1}$$

Robinson's [20] simplification of Hilbert's construction

$$R(x,y,z) = x^6 + y^6 + z^6 - (x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4) + 3x^2y^2z^2; \tag{2}$$

and

$$S(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2, \quad \text{due to Choi and Lam [1, 2].} \tag{3}$$

It is not too difficult to consider $qM, qR, qS$ for $q(x,y,z) = a^2x^2 + b^2y^2 + c^2z^2$, and determine whether these are sos using the algorithm of [3] directly or its implementation in, e.g., [15].

Interestingly enough, these conditions are the same in each case: the forms are sos if and only if

$$2(a^2b^2 + a^2c^2 + b^2c^2) \geq a^4 + b^4 + c^4.$$

This expression factors rather neatly into:

$$(a+b+c)(a+b-c)(b+c-a)(c+a-b) \geq 0,$$

so if $a \geq b \geq c \geq 0$ without loss of generality, the only non-trivial condition is that $b+c \geq a$; that is, there is a (possibly degenerate) triangle with sides $a, b, c$. (Robinson [20, p. 273] has a superficially similar condition, but note that his multiplier is $ax^2 + by^2 + cz^2$.)

By specializing this result and scaling variables as in the proof of the theorem, we note that

$$(x^2+y^2+z^2)M(x,\lambda y,\lambda z),\ (x^2+y^2+z^2)R(x,\lambda y,\lambda z),\ (x^2+y^2+z^2)S(x,\lambda y,\lambda z)$$

are sos if and only if $0 \leq \lambda \leq 2$.

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