ON THE ABSENCE OF UNIFORM DENOMINATORS IN HILBERT’S 17TH PROBLEM

BRUCE REZNICK

Abstract. Hilbert showed that for most \((n, m)\) there exist psd forms \(p(x_1, \ldots, x_n)\) of degree \(m\) which cannot be written as a sum of squares of forms. His 17th problem asked whether, in this case, there exists a form \(h\) so that \(h^2 p\) is a sum of squares of forms; that is, \(p\) is a sum of squares of rational functions with denominator \(h\). We show that, for every such \((n, m)\) there does not exist a single form \(h\) which serves in this way as a denominator for every psd \(p(x_1, \ldots, x_n)\) of degree \(m\).

1. Introduction

Let \(H_d(\mathbb{R}^n)\) denote the set of real homogeneous forms of degree \(d\) in \(n\) variables (“\(n\)-ary \(d\)-ics”). By identifying \(p \in H_d(\mathbb{R}^n)\) with the \(N = \binom{n+d-1}{n-1}\)-tuple of its coefficients, we see that \(H_d(\mathbb{R}^n) \approx \mathbb{R}^N\). Suppose \(m\) is an even integer. A form \(p \in H_m(\mathbb{R}^n)\) is called positive semidefinite or psd if \(p(x_1, \ldots, x_n) \geq 0\) for all \((x_1, \ldots, x_n) \in \mathbb{R}^n\). Following [1], we denote the set of psd forms in \(H_m(\mathbb{R}^n)\) by \(P_{n,m}\). Since \(P_{n,m}\) is closed under addition and closed under multiplication by positive scalars, it is a convex cone. In fact, \(P_{n,m}\) is a closed convex cone: if \(p_1 \rightarrow p\) coefficient-wise, and each \(p_n\) is psd, then so is \(p\). A psd form is called positive definite or pd if \(p(x_1, \ldots, x_n) = 0\) implies \(x_j = 0\) for \(1 \leq j \leq n\). The pd \(n\)-ary \(m\)-ics are the interior of the cone \(P_{n,m}\).

A form \(p \in H_m(\mathbb{R}^n)\) is called a sum of squares or sos if it can be written as a sum of squares of polynomials; that is, \(p = \sum_k h_k^2\). It is easy to show in this case that each \(h_k \in H_{m/2}(\mathbb{R}^n)\). Again following [1], we denote the set of sos forms in \(H_m(\mathbb{R}^n)\) by \(S_{n,m}\). Clearly, \(S_{n,m}\) is a convex cone; less obviously, it is a closed cone, a result due to R. M. Robinson [20].

In light of the inclusion \(S_{n,m} \subseteq P_{n,m}\), let \(\Delta_{n,m} = P_{n,m} \setminus S_{n,m}\). It was well-known by the late 19th century that \(P_{2,m} = S_{m,2}\) when \(m = 2\) or \(n = 2\). In 1888, Hilbert proved [8] that \(\Sigma_{3,4} = P_{3,4}\); more specifically, every \(p \in P_{3,4}\) can be written as the sum of three squares of quadratic forms. (An elementary proof, with “five” squares is in [2, pp.16-17]; for modern expositions of Hilbert’s proof, see [24] and [21].) Hilbert also proved in [8] that the preceding are the only cases for which \(\Delta_{n,m} = \emptyset\). That is,
if $n \geq 3$ and $m \geq 6$ or $n \geq 4$ and $m \geq 4$, then there exist psd forms $n$-ary $m$-ics that are not sos.

In 1893, Hilbert [9] generalized his three-square result for $P_{3,4}$ to ternary forms of higher degree. Suppose $p \in P_{3,m}$ with $m \geq 6$. Then there exist $p_1 \in P_{3,m-4}$ and $h_{1k} \in H_{m-2}(\mathbb{R}^3)$, $1 \leq k \leq 3$, so that

$$p_1p = h_{11}^2 + h_{12}^2 + h_{13}^2.$$ 

(Hilbert's proof seems to be non-constructive, and lacks a modern exposition. In the very recent paper [10], de Klerk and Pasechnik discuss the implementation of an algorithm to find $p_1$ so that $p_1p$ is sos, though not necessarily as a sum of three squares. This paper uses Hilbert's result without giving an independent proof.)

If $m = 6$ or $8$, then $p_1$ is a sum of three squares of forms, and hence (as Landau later noted [11]), the four-square identity implies that $p_1^2p = p_1(p_1p)$ is the sum of four squares of forms. If $m \geq 10$, then the argument can be applied to $p_1$; there exists $p_2 \in P_{3,m-8}$ with $p_2p_1 = h_{21}^2 + h_{22}^2 + h_{23}^2$. Thus, if $m = 10$ or 12 (so that $P_{3,m-8} = \Sigma_{3,m-8}$), then $(p_1p_2)^2p = p_2(p_2p_1)(p_1p)$ is the sum of four squares of forms, An easy induction shows that there exists $q \in H_t(\mathbb{R}^3)$ with $t = \lceil \frac{(m-2)^2}{8} \rceil$ so that $q^2p$ is the sum of four squares of forms.

Hilbert's 17th Problem asked whether this generalizes to $n > 3$ variables; that is, if $p \in P_{n,m}$, must there exist some form $q$ so that $q^2p$ is sos? Artin proved that there must be, in a way that gives no information about $q$. Much more on the history of this subject can be found in the survey paper [19].

This discussion leads to two closely related questions. Suppose $p \in P_{n,m}$. Can we find a form $h$ such that $hp$ is sos? Can we find a form $q$ so that $q^2p$ is sos? If we've answered the second, we've answered the first. Conversely, if $p \neq 0$ is psd and $hp$ is sos, then $h$ is psd. But it needn't be sos; indeed, a trivial answer to the first question is to take $h = p$. Stengle proved [23] that if $p(x, y, z) = x^3 z^3 + (y^2 z - x^3 - z^2 x)^2$, then $p^{2s+1} \in \Delta_{3,6}(2s+1)$ for every integer $s$. That is, $p^{2s-1} \cdot p$ is sos, but $p^{2s} \cdot p$ is not. Choi and Lam showed [1] that for $S \in \Delta_{3,6}$ (see (3) below), the product $S(x, y, z)S(x, z, y)$ is actually sos.

The author gratefully acknowledges correspondence with Chip Delzell, Pablo Parrilo, Vicki Powers, Marie-Françoise Roy and Claus Scheiderer. Their suggestions have made this a better paper.

2. WHAT IS KNOWN ABOUT THE DENOMINATOR

The first concrete result about a denominator in Hilbert's 17th Problem was found by Pólya [16]. He showed that if $f \in H_d(\mathbb{R}^n)$ is positive on the unit simplex $\{(x_1, \ldots, x_n) \mid x_j \geq 0, \sum x_j = 1\}$, then for sufficiently large $N$, $(\sum x_j)^N f$ has positive coefficients. Replacing each $x_j$ by $x_j^2$, we see that if $p \in H_{2d}(\mathbb{R}^n)$ is an even positive definite form, then $(\sum x_j^2)^N p$ is a sum of even monomials with positive coefficients, and so, as it stands, is a sum of squares of monomials. Taking even $N$, we see that $q = (\sum x_j^2)^{N/2}$ is a denominator for $p$. Habicht [6] generalized Pólya's
proof to give an alternate solution to Hilbert’s 17th Problem for pd forms; however, 
h is not readily constructible and in general is no longer a power of \( \sum x_j^2 \). Except 
for one example, Pólya did not attempt to determine an explicit value of \( N \). A good 
exposition of the theorems of Pólya and Habicht can be found in [7].

For positive definite \( p \in P_{n,m} \), let 
\[
\epsilon(p) := \frac{\inf \{ p(u) : u \in S^{n-1} \}}{\sup \{ p(u) : u \in S^{n-1} \}}
\]
measure how “close” \( p \) is to having a zero. The author [18] showed that if 
\[
N \geq \frac{nm(m - 1)}{(4 \log 2) \epsilon(p)} - \frac{n + m}{2},
\]
then \( (\sum_j x_j^2)^N p \) is a sum of \( (m + 2N) \)-th powers of linear forms, and so is sos. A 
similar lower bound has been shown to apply in Pólya’s case, one which goes to 
infinity as \( p \) approaches the boundary of \( P_{n,m} \). (See papers by de Loera and Santos 
[12] and by Powers and the author [17].)

The restriction to positive definite forms is necessary. There exist psd forms \( p \) in 
\( n \geq 4 \) variables so that, if \( h^2 p \) is sos, then \( h \) must have a specified zero. The existence 
of these unavoidable singularities, or so-called “bad points”, insures that \( (\sum x_j^2)^r p \) 
can never be a sum of squares of forms for any \( r \). Habicht’s Theorem implies that no 
positive definite form can have a bad point. Bad points were first noted by Straus 
and have been extensively studied by Delzell; see, e.g. [4, 5].

3. Recent results and a new theorem

Scheiderer has shown in very recent work [22] that for \( p \in P_{3,m} \), there exists 
\( N = N(p) \) so that \( (x^2 + y^2 + z^2)^N p(x, y, z) \) is sos; indeed, \( x^2 + y^2 + z^2 \) can be replaced 
by any positive definite form. This is a strong refutation to the existence of bad 
points for ternary forms.

Also very recently, Lombardi and Roy [13] have constructed a quantitative version 
of the Positivstellensatz. A special case is that for fixed \( (n, m) \), there exists \( d = d(n, m) \) so that if \( p \in P_{n,m} \), there exists \( q \in H_d(\mathbb{R}^n) \) so that \( q^2 p \) is sos.

Suppose \( (n, m) \) is such that \( \Delta_{n,m} \neq 0 \). Theorem 1 below states that there is no 
single form \( h \) so that, if \( p \in P_{n,m} \), then \( hp \) is sos. Corollary 2 says that there is not 
ev en a finite set of forms \( \mathcal{H} \) so that, if \( p \in P_{n,m} \), then there exists \( h \in \mathcal{H} \) so that \( hp \) 
in sos. In particular, there does not exist a finite set of denominators which apply 
to all of \( P_{n,m} \). This result implies that \( N(p) \) in Scheiderer’s theorem is not bounded 
as \( p \) ranges over \( P_{3,m} \). It also implies that the denominators in the Lombardi-Roy 
theorem cannot be chosen from a finite, predetermined set.

The proof of the Theorem is elementary and relies on a few simple observations.
If \( p \neq 0 \) is psd and \( hp \) is sos, then \( h \) is psd. As previously noted, \( \Sigma_{n,m} \) is a closed 
cone for all \( (n, m) \). This cone is invariant under the action of taking invertible linear 
changes of form. Thus, if \( h' \) is derived from \( h \) by such a linear change, and if \( hp \) is sos
for every \( p \in P_{n,m} \), then so is \( h'p \). Suppose \( \ell \) is a linear form, \( p = \sum_j g_j^2 \) is sos, and \( \ell \mid p \). Then \( \ell^2 \mid p \) and \( \ell \mid g_k \) for each \( k \), and by induction, \( \ell^{2s} \mid p \implies \ell^s \mid g_k \). Thus, we can “peel off” squares of linear factors from any sos form; this is a common practice, dating back at least to [20, p. 267]. We use this observation in the contrapositive: if \( p \in \Delta_{n,m} \), then \( \ell^{2s} p \in \Delta_{n,m+2s} \).

**Theorem 1.** Suppose \( \Delta_{n,m} \neq \emptyset \). Then there does not exist a non-zero form \( h \) so that if \( p \in P_{n,m} \), then \( hp \) is sos.

**Proof.** Suppose to the contrary that such a form \( h \) exists. Since \( h \neq 0 \), there exists a point \( a \in \mathbb{R}^n \) so that \( h(a) \neq 0 \). By making an invertible linear change of variables, we can take \( a = (1,0,\ldots,0) \). Thus, we may assume without loss of generality that \( h(x_1,0,\ldots,0) = \alpha x_1^d \), where \( \alpha > 0 \) and \( d \) is even. In the sequel, we distinguish \( x_1 \) from the other variables.

Choose \( p \in P_{n,m} \setminus \Sigma_{n,m} \). Then

\[
h(x_1,x_2,\ldots,x_n)p(x_1,rx_2,\ldots,rx_n)
\]

is sos for every \( r \in \mathbb{N} \). By making the change of variables \( x_i \to x_i/r \) for \( i \geq 2 \), we see that

\[
h(x_1,r^{-1}x_2,\ldots,r^{-1}x_n)p(x_1,x_2,\ldots,x_n)
\]

is also sos. Since

\[
\lim_{r \to \infty} h(x_1,r^{-1}x_2,\ldots,r^{-1}x_n) = h(x_1,0,\ldots,0) = \alpha x_1^d,
\]

and since \( \Sigma_{n,m+d} \) is closed, it follows that

\[
\lim_{r \to \infty} h(x_1,r^{-1}x_2,\ldots,r^{-1}x_n)p(x_1,x_2,\ldots,x_n) = \alpha x_1^dp(x_1,\ldots,x_n)
\]

is sos. Thus \( p \) is sos, a contradiction. \( \square \)

The following elegant proof is due to Claus Scheiderer and is included with his permission; it supersedes the proof in an earlier version of this manuscript.

**Corollary 2.** Suppose \( \Delta_{n,m} \neq \emptyset \). Then there does not exist a finite set of non-zero forms \( \mathcal{H} = \{h_1,\ldots,h_N\} \) with the property that, if \( p \in P_{n,m} \), then \( h_k p \) is sos for some \( h_k \in \mathcal{H} \).

**Proof.** Suppose \( \mathcal{H} \) exists. For each \( k \), there exists non-zero \( p \in \Delta_{n,m} \) so that \( h_k p \) is sos. (Otherwise, we may delete \( h_k \) harmlessly from \( \mathcal{H} \).) Thus, each \( h_k \) is psd, and there exists a form \( q_k \) so that \( q_k^2 h_k \) is sos. Define \( h = \prod_k q_k^2 h_k \). We now show that for every \( p \in P_{n,m} \), \( hp \) is sos: this contradicts the Theorem and proves the Corollary. By hypothesis, there exists \( h_j \in \mathcal{H} \) so that \( h_j p \) is sos. Thus,

\[
hp = \left( \prod_{k \neq j} q_k^2 h_k \right) \cdot q_j^2 \cdot h_j p
\]

is a product of sos factors, and so is sos. \( \square \)
Finally, we know by Hilbert’s theorem that for \( p \in P_{3,6} \), there exists quadratic \( h \) so that \( hp \in \Sigma_{3,8} \). The three simplest forms in \( \Delta_{3,6} \) are
\[
(1) \quad M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2, \quad \text{due to Motzkin [14]};
\]

Robinson’s [20] simplification of Hilbert’s construction
\[
(2) \quad R(x, y, z) = x^6 + y^6 + z^6 - (x^4y^2 + x^2y^4 + x^4z^2 + y^4z^2 + y^2z^4) + 3x^2y^2z^2;
\]

and
\[
(3) \quad S(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2, \quad \text{due to Choi and Lam [1, 2]}.\]

It is not too difficult to consider \( qM, qR, qS \) for \( q(x, y, z) = a^2x^2 + b^2y^2 + c^2z^2 \), and determine whether these are sos using the algorithm of [3] directly or its implementation in, e.g., [15].

Interestingly enough, these conditions are the same in each case: the forms are sos if and only if
\[
2(a^2b^2 + a^2c^2 + b^2c^2) \geq a^4 + b^4 + c^4.
\]

This expression factors rather neatly into:
\[
(a + b + c)(a + b - c)(b + c - a)(c + a - b) \geq 0,
\]

so if \( a \geq b \geq c \geq 0 \) without loss of generality, the only non-trivial condition is that \( b + c \geq a \); that is, there is a (possibly degenerate) triangle with sides \( a, b, c \). (Robinson [20, p. 273] has a superficially similar condition, but note that his multiplier is \( ax^2 + by^2 + cz^2 \).)

By specializing this result and scaling variables as in the proof of the theorem, we note that
\[
(x^2 + y^2 + z^2)M(x, \lambda y, \lambda z), \quad (x^2 + y^2 + z^2)R(x, \lambda y, \lambda z), \quad (x^2 + y^2 + z^2)S(x, \lambda y, \lambda z)
\]

are sos if and only if \( 0 \leq \lambda \leq 2 \).

**References**


[22] Scheiderer, C., Sums of squares on compact real algebraic surfaces, in preparation.

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801
E-mail address: reznick@math.uiuc.edu