A walk down the arithmetic-geometric mean streets of mathematics

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Before I begin my presentation tonight, I want to mention that there are more than 100 undergraduate and graduate students from Singapore at my home institution, the University of Illinois at Urbana-Champaign. They are currently taking final exams, Urbana had 8 inches of snow last weekend and it’s -6C there right now.
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On behalf of the participants in our conference/workshop, I want to take this public opportunity to thank Ms. Eileen Tan and Ms. Nurleen Binte Muhamed of the IMS staff for their superb organizational skills. I also want to thank the other very helpful members of the staff whose names I haven’t learned. I have been to many conferences in many places and this has been (by far) the best one in preparing local information for those who come from far away.
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Mathematicians are comfortable with algebra, but maybe some of you aren’t, so I’d like to tell you what a few letters will always mean in this talk.

When I use letters at the end of the alphabet \((w, x, y, z)\), I will mean ordinary real numbers which are either positive or zero. When I don’t want to say exactly how many numbers I’m using, I will use subscripts: \(x_1, x_2, \ldots, x_n\).
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Whenever you think of letters in algebra, think of pronouns: “\(x\)” = “she”, “\(y\)” = “he”, etc. These have different meanings in different contexts, and “\(x^2 + 7y\)” is easier to say than “the first number squared plus 7 times the second number”.
The first average is the **arithmetic mean** and it’s always $A$:

$$x_1 + x_2 + \cdots + x_n$$

$$= A + A + \cdots + A$$

This is the usual average you’ve seen working with data or dollars or calculating grades in a class.
The first average is the **arithmetic mean** and it’s always $A$:

$$x_1 + x_2 + \cdots + x_n = A + A + \cdots + A$$
or

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = A$$

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This is the usual average you’ve seen working with data or dollars or calculating grades in a class.
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or

$$G^n = x_1 \times x_2 \times \cdots \times x_n$$

This average is less common. It can be used (for example) in determining the average rate of return on an investment over a period of time.

If any $x_i$ is equal to zero, then $G = 0$, no matter how big the others $x_i$’s are: If you lose all your money one year, it doesn’t matter how much you made the years before.
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If any $x_i$ is equal to zero, then $G = 0$, no matter how big the others $x_i$'s are: If you lose all your money one year, it doesn’t matter how much you made the years before.
And now, if you remember nothing else that I say, please remember this. It’s the biggest idea of the talk:

$$A \geq G$$

Moreover, $A = G \iff x_1 = x_2 = \cdots = x_n$. This fact, called the inequality of the arithmetic and geometric means or arithmetic-geometric inequality or AGI, will allow us to solve many problems which are ordinarily given in calculus classes, and you won’t need to use calculus. They are simple examples in the subject of mathematical optimization.
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I ought to remind you about some rules of exponents from high school algebra in case you haven’t used them in a while:

\[ x^a \times x^b = x^{a+b} \quad (x^a)^b = x^{ab}. \]

(These may look boring, but they have inspired an exciting new branch of mathematics called “tropical geometry”, which is one of the topics at our Workshop.)
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The AGI, in one line, is

\[ \frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{1/n}. \]

Multiplying both sides of this equation by \( n \) we get an equivalent expression:

\[ x_1 + x_2 + \cdots + x_n \geq n(x_1 x_2 \cdots x_n)^{1/n}. \]
We mathematicians like strange and non-obvious substitutions if they allow us to simplify expressions. For example, since $x_i \geq 0$, we can write $x_i = y_i^n$ for $y_i = \sqrt[n]{x_i}$. The reason to do this is that

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If you believe what I just said, the AGI for \( n = 2 \) is the same thing as the assertion that

\[
y_1^2 + y_2^2 \geq 2y_1y_2; \quad y_1^2 + y_2^2 = 2y_1y_2 \iff y_1 = y_2
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I want to \textbf{prove} that this is true. Those who are good at algebra can see what’s coming next:

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This last statement is true because of an old fact from school: if $t$ is any real number and $t \neq 0$, then $t^2 > 0$, whether $t$ is positive or negative. Of course $0^2 = 0$. Much of my own research in abstract algebra starts from the observation that squares can’t be negative.
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Bruce Reznick  
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*A walk down the arithmetic-geometric mean streets*
Proofs are extremely important to mathematicians! Mathematical statements such as the ones I’m giving today are not true because I’m a big shot professor and I say they’re true. (I’m not such a big shot anyway.) Mathematical statements are true because we can formulate a proof that follows the accepted rules. Mathematicians forgive mistakes in our colleagues (and ourselves) but we don’t forgive deliberately false argument. And, though students may not believe it, proofs were not invented to torture them.
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Mathematical truth does not depend on the cultural environment: \(1 + 1 = 2\) for boys and for girls; \(1 + 1 = 2\) in North Korea and in North Carolina. If there is intelligent life in space, then \(1 + 1 = 2\) in whatever language they use there.
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One tricky thing about proofs is that the correctness of the logic matters, but the argument might start with something that looks unrelated to the task at hand. Students (and professionals) often start to read a proof and wonder where it’s going for a while. Mathematicians have to justify that proofs are correct, but we don’t have to explain how we happened to find them.
The first historical proof of the AGI for $n = 2$ comes from Euclid’s *Elements*, and is more than 2000 years old.
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I’ll complete the proof on the next two slides with a smaller picture.
Draw line segments $AB$ and $BC$, $x = |AB| > |BC| = y$. Construct $E$, the midpoint of $AC$, and draw the circle through $A$ and $C$ with center $E$. Construct perpendiculars to $AC$ at $B$ and $E$, intersecting the circle at $D$ and $F$. $EF$ is a radius, so $|EF| = \frac{1}{2} |AC| = x + y^2$. Observe that $BD$ is a side of a right triangle with hypotenuse $ED$, so $|BD| < |ED| = |EF| = x + y^2$. Since $ED$ is a radius, we have $|BD| < |ED| = |EF| = x + y^2$.
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$$|BD| < |ED| = |EF| = \frac{x+y}{2}$$
The triangles $ABD$ and $DBC$ are similar, since $\angle ADC$ is right. Thus, $|AB|/|BD| = |BD|/|BC| \implies |BD|^2 = xy$. In case $x = y$, $B$ coincides with $E$ and $D$ coincides with $F$ and there's nothing to prove.
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\[
\frac{|AB|}{|BD|} = \frac{|BD|}{|BC|} \quad \Rightarrow \quad \frac{x}{|BD|} = \frac{|BD|}{y} \quad \Rightarrow
\]

\[
\sqrt{xy} = |BD| < |ED| = \frac{x + y}{2}.
\]

In case $x = y$, B coincides with E and D coincides with F and there’s nothing to prove.
Here’s one calculus problem which can be solved directly by the AGI. Find the proportions of a rectangle of perimeter 12” which has *largest* area. Say this rectangle has height $x$ and width $y$. Then perimeter $= 2x + 2y$ and area $= x \times y$. Using only integers, we could have $x = 1, y = 5$, area 5; or $x = 2, y = 4$, area 8; or $x = 3, y = 3$, area 9.

What the AGI tells us is that $x + y \geq \sqrt{xy}$, neither of these expressions are exactly what we have, but $x + y$ equals half the perimeter, so $x + y$ is one quarter, and $\frac{1}{4} \times$ perimeter $\geq \sqrt{\text{area}} \iff$ perimeter $\geq 4\sqrt{\text{area}} \iff$ perimeter $^2 \geq 16 \times \text{area}$. Remember this for the next page.
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$$\frac{1}{4} \times \text{perimeter} \geq \sqrt{\text{area}} \iff \text{perimeter} \geq 4\sqrt{\text{area}}$$

$$\iff \text{perimeter}^2 \geq 16 \times \text{area}.$$  

Remember this for the next page.
Since we’re given that the perimeter is 12, the AGI tells us that

\[ \text{perimeter}^2 \geq 16 \times \text{area} \implies 12^2 \geq 16 \times \text{area} \]

\[ \implies \frac{12^2}{16} = 9 \geq \text{area}. \]

No matter how we choose the sides of the rectangle, the area is at most 9. And, the only way we get exactly 9 is to choose \( x = y \) so that the rectangle is a square. Since \( x = y \) and \( 2x + 2y = 12 \), we have \( x = y = 3 \).
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If you think about it, we have simultaneously solved a different problem as well. Suppose we have a rectangle with a fixed area. What is the \textit{smallest} perimeter? Looking at \( \text{perimeter} \geq 4\sqrt{\text{area}} \), we see a minimum value for the perimeter, and again it comes when \( x = y \).
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Largest perimeter? Forget it. Make a really long and skinny rectangle. There’s no largest perimeter.
Here’s another a standard calculus problem with a graphic borrowed from the Web.

You want to enclose a rectangular garden with one side against a barn. What proportions give the maximum area?

This time perimeter = 2W + L (because the side of the barn is “free”) and area = W × L. We can apply the AGM just as we did before, but with different letters. Use x = 2W and y = L:

\[
\text{perimeter} = 2W + L \geq \sqrt{(2W) \times L} = \sqrt{2 \times \text{area}}
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The maximum value occurs only when 2W = L. This makes sense.

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Bruce Reznick  University of Illinois at Urbana-Champaign  A walk down the arithmetic-geometric mean streets
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\frac{\text{perimeter}}{2} = \frac{2W + L}{2} \geq \sqrt{(2W) \times L} = \sqrt{2 \times (WL)} = \sqrt{2 \times \text{area}}
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The maximum value occurs only when \(2W = L\). This makes sense. One part of the length is “free”, so you can use more of it.
Time for some more proofs and then more examples. What happens if $n = 3$? Remember: proofs have to be true, not obvious. I first want to add up three squares:

$$(x - y)^2 + (x - z)^2 + (y - z)^2 =$$

$$(x^2 - 2xy + y^2) + (x^2 - 2xz + z^2) + (y^2 - 2yz + z^2) =$$

$$2x^2 + 2y^2 + 2z^2 - 2xy - 2xz - 2yz =$$

$$2(x^2 + y^2 + z^2 - xy - xz - yz)$$
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2x^2 + 2y^2 + 2z^2 - 2xy - 2xz - 2yz = \\
2(x^2 + y^2 + z^2 - xy - xz - yz)
\]

It follows that \( x^2 + y^2 + z^2 - xy - xz - yz \geq 0 \) and the only way it equals zero is if \( x - y = x - z = y - z = 0 \); that is, \( x = y = z \).

Suppose \( x, y, z \) are all \( \geq 0 \), so \( x + y + z \geq 0 \). It just happens to be an algebraic fact that

\[
(x^2 + y^2 + z^2 - xy - xz - yz)(x + y + z) = x^3 + y^3 + z^3 - 3xyz.
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Time for some more proofs and then more examples. What happens if \( n = 3 \)? Remember: proofs have to be true, not obvious.

I first want to add up three squares:

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You should check the algebra, for homework.
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$$(x^2 - 2xy + y^2) + (x^2 - 2xz + z^2) + (y^2 - 2yz + z^2) =$$

$$2x^2 + 2y^2 + 2z^2 - 2xy - 2xz - 2yz =$$

$$2(x^2 + y^2 + z^2 - xy - xz - yz)$$

It follows that $x^2 + y^2 + z^2 - xy - xz - yz \geq 0$ and the only way it equals zero is if $x - y = x - z = y - z = 0$; that is, $x = y = z$.

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$$(x^2 + y^2 + z^2 - xy - xz - yz)(x + y + z) = x^3 + y^3 + z^3 - 3xyz.$$

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You weren't expecting homework, were you?!
On the last page, I showed that \( x^3 + y^3 + z^3 - 3xyz \geq 0 \), so
\( x^3 + y^3 + z^3 \geq 3xyz \). If you remember from a while back, this is
the statement of the AGI for \( n = 3 \). Maybe you don’t like this
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$$\frac{x + y + z + w}{4} \geq \sqrt[4]{xyzw}.$$ 

I want to re-write the left-hand side and then apply the AGI for
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Terse and maybe unmotivated, but every step is true, and if you check where it’s equal, you’ll find that $x = y = z = w$. 

Bruce Reznick University of Illinois at Urbana-Champaign 

A walk down the arithmetic-geometric mean streets
It turns out that you can always go from the AGI with $n$ numbers to the AGI with $2n$ numbers, but that doesn’t help if you have an odd number of numbers. But there’s one more computational technique, which lets you go from $n$ to $n - 1$. I’ll just show it here when $n = 4$. We know that

$$\frac{x + y + z + w}{4} \geq (xyzw)^{1/4}.$$

for all $x, y, z, w$. 

Mathematicians love this kind of calculation!
It turns out that you can always go from the AGI with $n$ numbers to the AGI with $2n$ numbers, but that doesn’t help if you have an odd number of numbers. But there’s one more computational technique, which lets you go from $n$ to $n-1$. I’ll just show it here when $n = 4$. We know that

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(xyzw)^{1/4} = (xyz(xy)z)^{1/3}^{1/4} = ((xyz)^{4/3})^{1/4} = (xyz)^{1/3}
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Mathematicians love this kind of calculation!
I will tell my friends in the audience that AGI\(_n\) \(\rightarrow\) AGI\(_{2n}\) and AGI\(_n\) \(\rightarrow\) AGI\(_{n-1}\) and this is enough to prove the general statement by induction, with equality only when the variables are equal. For the non-mathematicians, please believe me that it can be proved correctly.

Let’s do some more problems. Given a rectangular box with sides \(x\), \(y\), \(z\) and a fixed surface area, how can you maximize the volume? The sides have area \(xy\), \(xz\) and \(yz\), and each one appears twice, so the surface area is \(2(xy + xz + yz)\) and the volume is \(xyz\). It’s pretty obvious by now what to do next:

\[
\text{surface area}^3 = 2(xy + xz + yz)^3 \\
\geq (2xy)^{1/3} (2xz)^{1/3} (2yz)^{1/3} = (8x^2y^2z^2)^{1/3} = 2^{2/3} \text{ volume}^{2/3}
\]

The upper bound on the volume occurs when \(2xy = 2xz = 2yz\), or \(x = y = z\). You could have guessed that.
I will tell my friends in the audience that $\text{AGI}_n \implies \text{AGI}_{2n}$ and $\text{AGI}_n \implies \text{AGI}_{n-1}$ and this is enough to prove the general statement by induction, with equality only when the variables are equal. For the non-mathematicians, please believe me that it can be proved correctly.

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The upper bound on the volume occurs when $2xy = 2xz = 2yz$, or $x = y = z$. You could have guessed that.
What happens though, if you have an open box, say one missing the lid? To be specific, the length is $x$, the width is $y$ and the height is $z$. The missing lid had area $xy$, so the actual surface area is $xy + 2xz + 2yz$. The volume is still $xyz$. Almost the same calculation applies:
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$$\frac{\text{surface area}}{3} = \frac{xy + 2xz + 2yz}{3} \geq ((xy)(2xz)(2yz))^{1/3}$$

$$= (4x^2y^2z^2)^{1/3} = 3\sqrt[3]{4(\text{volume})^{2/3}}$$

The maximum now occurs when $xy = 2xz = 2yz$, or $x = y = 2z$. Just as with the garden, because the cost of using $x$ and $y$ is less, they are larger than $z$. 
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A popular variation on this in calculus classes is to imagine boxes in which the tops and the various sides are made out of different materials with different costs, so the total cost is something like $axy + bxz + cyz$, where $a$, $b$, $c$ are given by the data in the problem. The method of solution is the same, and you never truly need calculus.
Something different. You are offered two different stocks: one will go up 5% a year for five years, and one will go up 0%, then 0%, then 5%, then 5%, then 15%. Notice that \( \frac{0+0+5+5+15}{5} = 5 \), so both “average” 5% a year.
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But which one did better? In financial problems, you want to multiply. Going up by \( i \)% is the same thing as multiplying the total by \( 1 + \frac{i}{100} \). Let’s say for simplicity that the stock started at $100.
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But which one did better? In financial problems, you want to multiply. Going up by \( i \)% is the same thing as multiplying the total by \( 1 + \frac{i}{100} \). Let’s say for simplicity that the stock started at $100. In the first case, you have

\[
$100 \times 1.05 \times 1.05 \times 1.05 \times 1.05 \times 1.05 = $127.63.
\]

The second case you have

\[
$100 \times 1.00 \times 1.00 \times 1.05 \times 1.05 \times 1.15 = $126.79.
\]

Not a huge difference, but fortunes have been made from less.
What is the true “average” growth in the second case? It’s really the geometric mean of 1.00, 1.00, 1.05, 1.05 and 1.15, which is 1.0486, or 4.86%. The AGI says that the true average rate of growth is never bigger than the average of the yearly growth percentages, and only equal when the growth is the same every year.
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Here is the exact formulation of how the AGI helps you make money:

$$\text{average growth} = \left( \left( 1 + \frac{i_1}{100} \right) \cdots \left( 1 + \frac{i_n}{100} \right) \right)^{1/n} \leq \frac{(1 + \frac{i_1}{100}) + \cdots + (1 + \frac{i_n}{100})}{n} = 1 + \left( \frac{\frac{i_1}{100} + \cdots + \frac{i_n}{100}}{n} \right)$$
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The true average rate of growth is never bigger than the average of the yearly growth percentages, and only equal when the growth is the same every year.
The last geometric example I want to give is the standard right circular cylinder. A can of Coke. If the can has radius $r$ and height $h$, then its volume is $\pi r^2 h$. There are two parts to the surface area: the top and bottom each have area $\pi r^2$, and if you unfold the part in the middle, it becomes a rectangle with height $h$ and width $2\pi r$. Thus the surface area equals $2 \times \pi r^2 + (2\pi r) \times h$. What does the AGI say?
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This is ... correct but not helpful, because $r^3 h$ isn’t naturally related to (not a function of) the volume, which is a multiple of $r^2 h$. What to do? Break up one of the summands into two pieces. Why? Because it works. This will be done on the next page. The underlying algebra is used in a familiar part of high school science.
\[
\frac{\text{surface area}}{3} = \frac{2\pi r^2 + \pi rh + \pi rh}{3} \geq \sqrt[3]{(2\pi r^2)(\pi rh)(\pi rh)}
\]

\[
= (2\pi^3 r^4 h^2)^{1/3} = 2^{1/3} \pi \left( \frac{\text{volume}}{\pi} \right)^{2/3} = (2\pi)^{1/3} (\text{volume})^{2/3}.
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The best value occurs with \(2\pi r^2 = \pi rh = \pi rh\), or \(h = 2r\). This cylinder would have the same height as diameter. Coca-Cola doesn’t do it this way. The top and bottom may cost more than the sides, and there’s also welding.
The best value occurs with $2\pi r^2 = \pi rh = \pi rh$, or $h = 2r$. This cylinder would have the same height as diameter. Coca-Cola doesn’t do it this way. The top and bottom may cost more than the sides, and there’s also welding.

The arithmetic needed to get the proper ratio is exactly the kind used in “balancing” a chemical equation. Ignore the constants and turn exponents into the number of atoms:

$$R_2 + 2 \text{ RH} \rightarrow 2 \text{ R}_2\text{H}$$

This method lets you solve lots of other calculus problems!
The last topic is still related to the AGI, but is completely different in spirit. When Sinai Robins invited me to give this talk, his only specific request was that it have some connection with our Conference/Workshop. I would like to tell you a story and I will keep it non-technical as long as I can, but chilis will eventually show up.

Earlier in the talk, I showed that a specific polynomial $p$ always satisfied the condition $p(x, y, z) \geq 0$ by writing it as a sum of squares of other polynomials:

$$x^2 + y^2 + z^2 - xy - xz - yz = 1,$$

$$\left(x^2 + (x - y)^2 + (y - z)^2\right),$$

It is not accidental that I could do this. For any non-negative polynomial $p$ of degree 2, one can always find a way to write it as a sum of squares. But other polynomials aren't so amenable.
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Earlier in the talk, I showed that a specific polynomial \( p \) always satisfied the condition \( p(x, y, z) \geq 0 \) by writing it as a sum of squares of other polynomials:

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x^2 + y^2 + z^2 - xy - xz - yz = \frac{1}{2} \left( (x - y)^2 + (x - z)^2 + (y - z)^2 \right)
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It is not accidental that I could do this. For any non-negative polynomial $p$ of degree 2, one can always find a way to write it as a sum of squares. But other polynomials aren’t so amenable.
David Hilbert (1862-1943)

In 1888, David Hilbert proved that there exists a polynomial that satisfies $p(x, y, z) \geq 0$ and which is not a sum of squares of polynomials. This led to the 17th of his famous 1900 list of 23 problems which he thought would attract the attention of mathematicians in the 20th century. (My attention was certainly attracted.) The funny thing is that Hilbert said how you could find $p$ but he didn’t actually write it down in detail, because it was very complicated. It took almost 80 years for the first such polynomial to be discovered.
Before I give this story, I should tell you why people care about this. It has always been a matter of importance in applied mathematics to determine whether a particular polynomial \( p(x_1, \ldots, x_n) \geq 0 \). For various reasons, it is important to know, not only that this is true but to give a “certificate” that it is true. You might think that if you checked it at a million random values, that would be good enough, but it isn’t. (In fact, there’s some question about how you can tell whether values are random.)

If the number of variables is large, a million isn’t enough: When there are 20 variables, there are \( 2^{20} = 1,048,576 \) choices of \((x_1, \ldots, x_{20})\) even if you only want to check the value of \( p \) when each \( x_j \) takes the value 0 or 1, calculations such as

\[
p(1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 0, 0, 1, 1)
\]

Hardly seems like enough points.
The determination, with proof, whether $p(x_1, \ldots, x_n) \geq 0$ (in the absence of additional information) is a very difficult and possibly time-consuming question. On the other hand, if you know that $p = \sum h_k^2$, then you know automatically that $p(x_1, \ldots, x_n) \geq 0$. Fortunately, this sits in a class of computational problems that goes under the title of “semi-definite programming”, and you can decide very quickly whether $p$ is a sum of squares.
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then you know automatically that \( p(x_1, \ldots, x_n) \geq 0 \). Fortunately, this sits in a class of computational problems that goes under the title of “semi-definite programming”, and you can decide very quickly whether \( p \) is a sum of squares.

So about 15 years ago, many applied mathematicians and engineers became very interested in this question of writing polynomials as sums of squares. One observation, and I’ll leave it at that, is that if

\[ p(x_1, \ldots, x_n) = \frac{q(x_1, \ldots, x_n)}{r(x_1, \ldots, x_n)} \]

and both \( q(x_1, \ldots, x_n) \geq 0 \) and \( r(x_1, \ldots, x_n) > 0 \), then you’re done.
In the early 1960s, Theodore Motzkin organized a seminar at UCLA on the topic of different proofs of the AGI. In the course of his work, he discovered a remarkable example, which was published in 1967. Let

\[ M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \]

Everybody who works with this calls it the Motzkin polynomial. It does what Hilbert said it should do. It's easy to prove that it only takes positive values.
Theodore Motzkin (1908-1970)

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Everybody who works with this calls it the Motzkin polynomial. It does what Hilbert said it should do. It’s easy to prove that it only takes positive values.
Observe that

\[
M(x, y, z) = \frac{x^4y^2 + x^2y^4 + z^6}{3} - \frac{x^2y^2z^2}{3} = \frac{x^4y^2 + x^2y^4 + z^6}{3} - ((x^4y^2)(x^2y^4)(z^6))^{1/3} \geq 0
\]

Here, the inequality follows from the AGI, applied to the three numbers \(x^4y^2, x^2y^4, z^6\).
Observe that

\[
\frac{M(x, y, z)}{3} = \frac{x^4 y^2 + x^2 y^4 + z^6}{3} - x^2 y^2 z^2 = \frac{x^4 y^2 + x^2 y^4 + z^6}{3} - ((x^4 y^2)(x^2 y^4)(z^6))^{1/3} \geq 0
\]

Here, the inequality follows from the AGI, applied to the three numbers \(x^4 y^2, x^2 y^4, z^6\).

The last thing I’m going to talk about is an explanation of why the Motzkin polynomial is not a sum of squares of polynomials. The easy part is that since \(M(x, y, z)\) has degree 6, if it a sum of squares, it will be a sum of squares of polynomials of degree three. On the next page, I’m going to write a polynomial of degree three in \(x, y, z\) in a particular way, so that the term related to \(x^i y^j z^{3-i-j}\) appears at the point whose cartesian coordinates are \((i, j)\).
Suppose

\[ \sum_{m} q_m^2(x, y, z) = M(x, y, z) = +x^2y^4 -3x^2y^2z^2 +x^4y^2 +z^6, \]

where

\[ q_m(x, y, z) = \begin{pmatrix} a_m y^3 \\ +b_m y^2z \\ +c_m xy^2 \\ +d_m yz^2 \\ +e_m xyz \\ +f_m x^2y \\ +g_m z^3 \\ +h_m xz^2 \\ +i_m x^2z \\ +j_m x^3 \end{pmatrix}^2 \]

Then looking at the coefficient of \( y^6 \), we get \( 0 = \sum a_m^2 \), so \( a_m = 0 \), and similarly, looking at \( y^4z^2, y^2z^4, x^6, x^4z^2 \) and \( x^2z^4 \) in order, we get \( b_m = d_m = h_m = i_m = j_m = 0 \). This doesn’t leave a lot of terms left. (The fact that the shapes of the terms left in each case are the same kind of triangle is not an accident.)
The terms that are left in the squares have the same *shape* as the terms in the Motzkin polynomial.
Finally, we are left with

\[ x^4 y^2 + x^2 y^4 + z^6 - 3x^2 y^2 z^2 = \sum_{m} (f_m x^2 y + c_m x y^2 + g_m z^3 + e_m x y z)^2. \]

A computation of the coefficients of \( x^2 y^2 z^2 \) on both sides yields:

\[ -3 = \sum_{m} e_m^2. \]

This is a contradiction, because a sum of squares can’t be negative. This is Motzkin’s original proof, and to a mathematician it is a beautiful proof. Even if you are not a mathematician, I hope you see some of its great unexpectedness and elegance.
The title of this talk is based on a quote from one of my favorite mystery writers, Raymond Chandler, creator of Philip Marlowe:

*Down these mean streets a man must go who is not himself mean, who is neither tarnished nor afraid.*
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Humphrey Bogart as Philip Marlowe in *The Big Sleep*
Thank you!