RAMANUJAN MEMORIAL LECTURE

Some of the roots of Ramanujan's mathematics

Bruce Reznick

University of Illinois
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This talk is dedicated to the memory and mathematics of Srinivasa Ramanujan (1887-1920).

This talk is also dedicated to all the participants in the 2021 Ramanujan Contest.

I want to share an old saying from my student days. The slang may sound a little old-fashioned, but you can google the word after my talk.

Math is groovy!

Ramanujan's First Letter to Hardy, 1913

I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 23 years of age. I have had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. . . .

I would request you to go through the enclosed papers. Being poor, if your are convinced that there is anything of value I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get but I have indicated the lines on which I proceed. Being inexperienced I would very highly value any advice you give me. Requesting to be excused for the trouble I give you.

I remain, Dear Sir, Yours truly,

S. Ramanujan

G. H. Hardy later described his reaction to the final equations in Ramanujan's letter.

> *They defeated me completely. I had never seen anything in the least like them before.*
> *A single look at them is enough to show that they could only have been written by a mathematician of the highest class.*
> *They must be true because, if they were not true, no one could have had the imagination to invent them.*

This is the way most mathematicians feel about Ramanujan. My colleague Bruce Berndt has told me that in the roughly 4000 mathematical assertions that Ramanujan left us, apart from obvious typos, there are fewer than ten actual errors.

Ramanujan's equations are sometimes hard to understand, even for research mathematicians. Today, I want to present some of his amazing numerical identities, then step back to show their mathematical context, and return to the amazing Ramanujan and his seemingly instant insights.

Ramanujan loved equations involving square roots and cube roots (and higher roots) of integers. Here are a few parts of Problems he submitted to the *Journal of the Indian Mathematical Society*. The funny thing is that they look impossible, but they are actually easy to prove. (I'll show you one in a minute.)

Ramanujan's brilliance is in the discovery, not the proof. More specifically, what is profound about his work is the imagination to think that there might even *exist* a beautiful formula of the kind that he was able to discover repeatedly. What I will be talking about today are not his deepest mathematical insights, but a few relatively accessible results, which I hope will give you some of the flavor of his work.

This is just a sampling of some of the identities that Ramanujan posed as problems.

$$3 \cdot \sqrt{\sqrt[3]{5} - \sqrt[3]{4}} = \sqrt[3]{2} + \sqrt[3]{20} - \sqrt[3]{25},$$

$$3 \cdot \sqrt{\sqrt[3]{28} - 3} = \sqrt[3]{98} - \sqrt[3]{28} - 1,$$

$$\sqrt{\sqrt[5]{\frac{1}{5}} + \sqrt[5]{\frac{4}{5}}} = \sqrt[5]{1 + \sqrt[5]{2} + \sqrt[5]{8}},$$

$$\sqrt[6]{7\sqrt[3]{20} - 19} = \sqrt[3]{\frac{5}{3}} - \sqrt[3]{\frac{2}{3}}.$$

I'll prove the first equation for you, using only school algebra:

$$3 \cdot \sqrt{\sqrt[3]{5} - \sqrt[3]{4}} = \sqrt[3]{2} + \sqrt[3]{20} - \sqrt[3]{25}.$$

It's easy enough to square the left-hand side:

$$\left(3 \cdot \sqrt{\sqrt[3]{5} - \sqrt[3]{4}}\right)^2 = 3^2 \left(\sqrt{\sqrt[3]{5} - \sqrt[3]{4}}\right)^2 = 9 \left(\sqrt[3]{5} - \sqrt[3]{4}\right).$$

Now, I need to remind you of some school algebra: $\sqrt[3]{x}\sqrt[3]{y} = \sqrt[3]{xy}$, $\sqrt[3]{a^3 x} = a\sqrt[3]{x}$ (you can pull out cubes), and

$$(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc.$$

This will be expanded this with $a = \sqrt[3]{2}$, $b = \sqrt[3]{20}$, $c = \sqrt[3]{25}$ in the next frame show what happens when you square the right-hand side.

Remember that after squaring, what we want to show is that

$$9\left(\sqrt[3]{5}-\sqrt[3]{4}\right)=(\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25})^2.$$

But the right-hand side becomes, by the squaring rules and the others I mentioned:

$$(\sqrt[3]{2})^2+(\sqrt[3]{20})^2+(\sqrt[3]{25})^2+2\sqrt[3]{2}\sqrt[3]{20}-2\sqrt[3]{2}\sqrt[3]{25}-2\sqrt[3]{20}\sqrt[3]{25}$$
$$=\sqrt[3]{4}+\sqrt[3]{400}+\sqrt[3]{625}+2\sqrt[3]{40}-2\sqrt[3]{50}-2\sqrt[3]{500}.$$

But now we can pull out some cubes: $400=2^3\cdot 50$, $625=5^3\cdot 5$, $500=5^3\cdot 4$, and if we use $\sqrt[3]{a^3x}=a\sqrt[3]{x}$ and combine terms, this sum miraculously simplifies:

$$\sqrt[3]{4}+\sqrt[3]{400}+\sqrt[3]{625}+2\sqrt[3]{40}-2\sqrt[3]{50}-2\sqrt[3]{500}$$
$$=\sqrt[3]{4}+2\sqrt[3]{50}+5\sqrt[3]{5}+4\sqrt[3]{5}-2\sqrt[3]{50}-10\sqrt[3]{4}$$
$$(5+4)\sqrt[3]{5}+(2-2)\sqrt[3]{50}+(1-10)\sqrt[3]{4}=9\sqrt[3]{5}-9\sqrt[3]{4}. \qquad \checkmark$$

The others can be proved in a similar way. Not hard if you can keep track of the arithmetic. You can try to find generalizations with other numbers, but Ramanujan always gave us the "best" ones already. I'll show you with the first one.

What do I mean by that? Suppose you want to write out

$$(a+b-c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$$

and have two terms cancel out. The easiest way to do this is to have $b^2 = 2ac$. If you want to do this with cube roots, so $a = \sqrt[3]{r}$, $b = \sqrt[3]{s}$ and $c = \sqrt[3]{t}$ with integers r, s, t, then

$$b^2 = 2ac \iff \sqrt[3]{s^2} = 2\sqrt[3]{r}\sqrt[3]{t} \iff s^2 = 8rt.$$

The solution to $s^2 = 8rt$, up to multiples or switching r and t is:

$$r = 2u^2, \quad t = w^2, \quad s^2 = 16u^2w^2 \iff s = 4uw$$
$$\implies a = \sqrt[3]{2u^2}, \quad b = \sqrt[3]{4uw}, \quad c = \sqrt[3]{w^2}.$$

Notice that if $u = 1$ and $w = 5$, we get back Ramanujan's example with $2 \cdot 1^2 = 2$, $4 \cdot 1 \cdot 5 = 20$ and $5^2 = 25$!

Let's calculate. I'll use the same computational shortcuts I used before with pulling out cubes and factors of two:

$$
\begin{aligned}
&(\sqrt[3]{2u^2} + \sqrt[3]{4uw} - \sqrt[3]{w^2})^2 \\
&= \sqrt[3]{4u^4} + \sqrt[3]{16u^2w^2} + \sqrt[3]{w^4} + 2\sqrt[3]{8u^3w} - 2\sqrt[3]{2u^2w^2} - 2\sqrt[3]{8uw^3} \\
&= u\sqrt[3]{4u} + 2\sqrt[3]{2u^2w^2} + w\sqrt[3]{w} + 2\cdot 2u\sqrt[3]{w} - 2\sqrt[3]{2u^2w^2} - 2w\sqrt[3]{4u} \\
&= (4u+w)\sqrt[3]{w} - (2w-u)\sqrt[3]{4u} \implies \\
&\sqrt{(4u+w)\sqrt[3]{w} - (2w-u)\sqrt[3]{4u}} = \sqrt[3]{2u^2} + \sqrt[3]{4uw} - \sqrt[3]{w^2}.
\end{aligned}
$$

If we want to get a nice symmetric answer, then we want

$$
\begin{aligned}
&4u + w = 2w - u \iff 5u = w \implies \\
&\sqrt{9u(\sqrt[3]{5u} - \sqrt[3]{4u})} = \sqrt[3]{2u^2} + \sqrt[3]{20u^2} - \sqrt[3]{25u^2}.
\end{aligned}
$$

If we divide by the common factor of $u^{2/3}$, or take $u = 1$, we see that the *only* beautiful formula of this particular type is the one that Ramanujan already discovered!

I want to try to put this in context. First of all, why do we talk about numbers like $\sqrt[3]{4}$ or $\sqrt[3]{5}$, instead of expressing them as rational numbers?

Here is an argument that goes back 2500 years to Euclid's *Elements*. As far as one can tell, the proof has been told in the same way for close to 2500 years. There are very few sentences that have been spoken in the same way for 2500 years that aren't prayers or psalms of some kind.

And this proof *does* have a spiritual nature attached to it. Mathematicians love telling it to classes, even when it's not completely necessary. And we have been known to recite the proof to ourselves when we are alone. It can be very comforting!

One more recent bit of notation. Mathematicians use the terms $\mathbb{Z}$ and $\mathbb{Q}$ to represent the integers and the rational numbers, which are the ratios of integers.

Suppose

$$\sqrt{2} = \frac{a}{b}, \quad a, b \in \mathbb{Z} \implies b\sqrt{2} = a,$$

and b is as small as possible. Then by squaring, we get $2b^2 = a^2$.
This implies that a^2 is even, so a is even, and $a = 2c$ for some integer c,
But then $2b^2 = (2c)^2 = 4c^2 \implies b^2 = 2c^2$.
Thus, b^2 is even, and so b is even and $b = 2d$ for some integer d.
But then

$$\sqrt{2} = \frac{a}{b} = \frac{2c}{2d} = \frac{c}{d},$$

and $d = \frac{b}{2} < b$, which contradicts the minimality of b!
Therefore, the original equation can't happen and $\sqrt{2} \notin \mathbb{Q}$ is an irrational number after all.

In the same way, mathematicians can show that if m and n are integers, then either $\sqrt[m]{n}$ is an integer, or it is irrational. But it's still a number, so mathematicians use it in and of itself. In fact, we throw it in with $\mathbb{Q}$ and see what we get. I'll do this with $\sqrt{2}$ and then leave you with a formula Ramanujan could have given in his sleep. This is a well-known object in mathematics:

$$\mathbb{Q}(\sqrt{2}) = \left\{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \right\}.$$

This is the set of all rational numbers plus rational multiples of $\sqrt{2}$. You can do all the arithmetic you'd expect:

$$\begin{aligned}(a + b\sqrt{2}) \pm (c + d\sqrt{2}) &= (a + c) \pm (b + d)\sqrt{2}, \\ (a + b\sqrt{2}) \cdot (c + d\sqrt{2}) &= ac + bc\sqrt{2} + ad\sqrt{2} + bd(\sqrt{2})^2 \\ &= (ac + 2bd) + (bc + ad)\sqrt{2}.\end{aligned}$$

You can even divide, using the technique you probably learned in school, since $(c + d\sqrt{2})(c - d\sqrt{2}) = c^2 - 2d^2 \neq 0$, so

$$\frac{a + b\sqrt{2}}{c + d\sqrt{2}} = \frac{(a + b\sqrt{2})(c - d\sqrt{2})}{(c + d\sqrt{2})(c - d\sqrt{2})} = \frac{(ac - 2bd) + (bc - ad)\sqrt{2}}{c^2 - 2d^2}.$$

Can we take square roots in $\mathbb{Q}(\sqrt{2})$? Sometimes we can:

$$(1 + \sqrt{2})^2 = 1 + 2\sqrt{2} + (\sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}$$
$$\implies \sqrt{3 + 2\sqrt{2}} = 1 + \sqrt{2}.$$

Is there a way to try to solve for $\sqrt{3 + 2\sqrt{2}}$ without knowing the answer? Yes, but the method doesn't work well most of the time.

Here is the method: suppose a and b are rational.

$$\sqrt{3+2\sqrt{2}} = a + b\sqrt{2} \implies$$
$$3 + 2\sqrt{2} = (a + b\sqrt{2})^2 = a^2 + 2b^2 + 2ab\sqrt{2}$$

Since a and b are rational, we can separate this equation into two: $3 = a^2 + 2b^2, \; 2 = 2ab$. It follows that

$$a^2 + 2b^2 - 3ab = a^2 + 2b^2 - \tfrac{3}{2} \cdot 2ab = 3 - \tfrac{3}{2} \cdot 2 = 0,$$

and $a^2 + 2b^2 - 3ab = (a - b)(a - 2b)$, as you may remember from school. If $a = b$ then $2ab = 2$ implies that $2a^2 = 2$ or $a^2 = 1$, so $a = 1$ and $b = 1$, and we get the expected $1 + \sqrt{2}$.
By the way, if you try $a = 2b$, then $2ab = 2$ implies $4b^2 = 2$, so $b = \frac{1}{\sqrt{2}}$ and $a = 2b = \frac{2}{\sqrt{2}} = \sqrt{2}$, and $a + b\sqrt{2} = \sqrt{2} + \frac{1}{\sqrt{2}} \cdot \sqrt{2}$ $= \sqrt{2} + 1$, which is the same answer, which is funny, because a and b are not rational here!

The reason this doesn't work in general is that the quadratic equation usually doesn't factor. Then, you get some weird formulas, which are not Ramanujan-worthy to be sure, but are checkable by squaring. For example,

$$2\sqrt{3+\sqrt{2}} = \sqrt{6+2\sqrt{7}} + \sqrt{6-2\sqrt{7}}.$$

You could probably win a bar bet on this one!
The general formula is

$$2\sqrt{a+\sqrt{n}} = \sqrt{2a+2\sqrt{a^2-n}} + \sqrt{2a-2\sqrt{a^2-n}}.$$

To really understand equations like this, you need to use complex numbers and you need to study Galois theory, a branch of algebra. The French mathematician Evariste Galois (1811-1832), was another mathematical genius who accomplished a great deal in a very short life.

I'm going to get back to Ramanujan in a bit, but I first wanted to give you the background he knew as a basis of a remarkable anecdote, the Strand problem.

If $\sqrt{2}$ isn't rational, can we still find a rational number that is close to it? Notice that

$$\frac{a}{b} \approx \sqrt{2} \implies \left(\frac{a}{b}\right)^2 \approx (\sqrt{2})^2 \implies \frac{a^2}{b^2} \approx 2.$$

This suggests that a^2 is very close to $2b^2$. They're integers and we just saw that they can't be equal, so the closest they could be is to differ by 1. This brings up the question: Can we find positive integers a and b so that $a^2 - 2b^2 = 1$ or $a^2 - 2b^2 = -1$?

Sure! Here are all the solutions when $0 < a < 1000$:

$$1^2 - 2 \cdot 0^2 = 1, \quad 1^2 - 2 \cdot 1^2 = -1, \quad 3^2 - 2 \cdot 2^2 = 1,$$
$$7^2 - 2 \cdot 5^2 = -1, \quad 17^2 - 2 \cdot 12^2 = 1, \quad 41^2 - 2 \cdot 29^2 = -1,$$
$$99^2 - 2 \cdot 70^2 = 1, \quad 239^2 - 2 \cdot 169^2 = -1, \quad 577^2 - 2 \cdot 408^2 = 1.$$

The last fraction gives a very good approximation to $\sqrt{2}$.

$$\sqrt{2} \approx 1.4142136, \ \frac{577}{408} \approx 1.4142157, \ \left(\frac{577}{408}\right)^2 = 2 + \frac{1}{(408)^2}.$$

This is a special case of what is now known as the Pell Equation.

$$a^2 - Nb^2 = \pm 1.$$

The roots of this problem are ancient. The equation $a^2 - 2b^2 = \pm 1$ was studied 2400 years ago by the Greek Pythagoreans and also by Indian mathematicians over the centuries. For example, Brahmagupta studied the equation $a^2 - 92b^2 = 1$ in the year 628.

The Pell Equation has inspired a considerable mathematical literature. It turns out that there is an easy way to solve $a^2 - 2b^2 = \pm 1$, and here it is. Expand

$$(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}.$$

Then $a_n^2 - 2b_n^2 = (-1)^n$ takes the values ± 1 alternately, and these are the *only* solutions in positive integers.

We've already seen that $(1+\sqrt{2})^2 = 3+2\sqrt{2}$, and for example,

$$(1+\sqrt{2})^4 = (3+2\sqrt{2})^2 = 17+12\sqrt{2},$$
$$(1+\sqrt{2})^8 = (17+12\sqrt{2})^2 = 577+408\sqrt{2}.$$

Because $a_{n+1} + b_{n+1}\sqrt{2} = (1+\sqrt{2})^{n+1} = (1+\sqrt{2})(1+\sqrt{2})^n = (1+\sqrt{2})(a_n + b_n\sqrt{2})$, we also have formulas called recurrence relations:

$$a_{n+1} + b_{n+1}\sqrt{2} = (a_n + 2b_n) + (a_n + b_n)\sqrt{2}$$
$$\implies a_{n+1} = a_n + 2b_n, \qquad b_{n+1} = a_n + b_n.$$

A little more algebra tells you that

$$\frac{a_{n+1}}{b_{n+1}} = \frac{a_n + 2b_n}{a_n + b_n} = 1 + \frac{b_n}{a_n + b_n} = 1 + \frac{1}{\frac{a_n+b_n}{b_n}} = 1 + \frac{1}{1+\frac{a_n}{b_n}}$$

so it's easy to find a_n, b_n without multiplying out $(1+\sqrt{2})^n$.

Let me repeat that last formula and show it in action with $\frac{a_2}{b_2} = \frac{3}{2}$.

$$\frac{a_{n+1}}{b_{n+1}} = 1 + \frac{1}{1 + \frac{a_n}{b_n}}$$

$$\frac{a_3}{b_3} = 1 + \frac{1}{1 + \frac{a_2}{b_2}} = 1 + \frac{1}{1 + \frac{3}{2}} = 1 + \frac{1}{\frac{5}{2}} = 1 + \frac{2}{5} = \frac{7}{5}.$$

You can also use continued fractions, another subject at which Ramanujan was an amazing master:

$$\frac{a_2}{b_2} = \frac{3}{2} = 1 + \frac{1}{2},$$

$$\frac{a_3}{b_3} = \frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}},$$

$$\frac{a_4}{b_4} = \frac{17}{12} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}},$$

etc.

The full story is that, in a reasonably precise sense,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}.$$

This can be proved by adding 1 to both sides. Then,

$$x = 1 + \sqrt{2} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}} = 2 + \frac{1}{x}$$

$$\implies x = 2 + \frac{1}{x} \implies x^2 = 2x + 1$$

$$\implies x^2 - 2x - 1 = 0 \implies x = 1 \pm \sqrt{2}.$$

Since $x > 0$, we take the plus sign in the solution to the quadratic equation. Ramanujan worked with continued fractions in a technically more complicated way than I can explain in detail, but here is one of his many extraordinary formulas:

This beautiful formula combines e, π and the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2}$$

into a single amazing equation. Notice that in the continued fraction below, it is the numerators that change, not the denominators, and they are the increasing powers of $e^{-2\pi}$ (what an imagination to think of that!):

$$1 + \frac{e^{-2\pi}}{1 + \frac{e^{-4\pi}}{1+\frac{e^{-6\pi}}{1+\frac{e^{-8\pi}}{1+\cdots}}}} = \frac{e^{-2\pi/5}}{\sqrt{\phi\sqrt{5}} - \phi} \approx 1.0018764.$$

This is a single special case of the famous Rogers-Ramanujan identities.

I'd like to finish with a story from Robert Kanigel's great biography *The Man Who Knew Infinity*. This takes place in December 1914, during the early years of World War One, when Ramanujan was in his first winter in England working with Hardy.

One day, he was in his rooms stirring vegetables over a gas fire. His visitor was the statistician P. C. Mahalanobis, a student at King's College at the time, who read Ramanujan a riddle story from the *Strand* magazine, about a man who had a friend in the Belgian city of Louvain.

> *He said the house of a friend was on a long street, numbered one, two, three, and so on, and that all the numbers on one side of him added up exactly the same as all the numbers on the other side of him. Funny thing, that! He said there were more than fifty houses on that side of the street, but not so many as five hundred.*

The riddle was to find the number of the friend's house. Kanigel writes "Through trial and error ... Mahalanobis figured it out in a few minutes."

> *Ramanujan figured it out too, but with a twist: 'Please take down the solution', he said and proceeded to dictate a continued fraction. This wasn't just the solution to the problem, it was the solution to the whole class of problems implicit in the puzzle.*

Whether you know it or not, you've already seen you the answer! To clarify the question with a simple example, suppose the person lived in house 6 and there are 8 houses on the block. Then the lower numbered houses are:

$$1 + 2 + 3 + 4 + 5 = 15.$$

and the higher numbered houses are

$$7 + 8 = 15.$$

I'll explain the solution by starting with a basic, familiar formula for math majors. The triangular numbers T_r are defined by

$$T_r = 1 + 2 + ... + r = \frac{r(r+1)}{2}.$$

Rather than give a proof, I'll illustrate this with a famous story from the history of mathematics. The teacher of a class in rural Germany in the 1780s gave his students the task of finding $T = 1 + 2 + \cdots + 100$ as a way of keeping the class quiet for a half hour.

But the 8 year old Carl Friedrich Gauss, one of the greatest mathematicians who ever lived, happened to be in the class, and immediately saw the answer. This is one possible way. Write T twice, forwards and backwards:

$$\begin{aligned} T &= 1 \quad &+2+ \quad &\cdots + 99 \quad &+100 \\ T &= 100 \quad &+99+ \quad &\cdots + 2 \quad &+1 \\ 2T &= 101 \quad &+101+ \quad &\cdots + 101 \quad &+101 \end{aligned}$$

so $2T = 100 \cdot 101 \implies T = \frac{100 \cdot 101}{2}$.

Suppose the person lives in the house numbered n and suppose there are M houses on the block. Because the lower numbered houses stop at $n-1$, their sum is

$$1 + 2 + \cdots + n - 1 = T_{n-1} = \frac{(n-1)n}{2}.$$

You could figure out a formula the sum of the higher numbered houses, $(n+1) + ... + M$, but you don't have to! The condition of the problem are that it's the same sum as the lower houses

$$(n+1) + (n+2) + \cdots + M = \frac{(n-1)n}{2}.$$

We now know that

$$T_M = \frac{M(M+1)}{2} = 1 + \cdots + M$$
$$= (1 + \cdots + n - 1) + n + ((n+1) + (n+2) + \cdots + M)$$
$$= \frac{(n-1)n}{2} + n + \frac{(n-1)n}{2} = (n-1)n + n = n^2.$$

In other words, the triangular number $\frac{M(M+1)}{2}$ is also a square n^2. For the example I gave earlier, $T_8 = \frac{8 \cdot 9}{2} = 36 = 6^2$. There were 8 houses on the block, and his friend lived at 6.

I am confident that this is a problem that Ramanujan had already read about or solved; it's a well-known question in Diophantine analysis that was already studied by Euler in the late 18th century. It even has its own Wikipedia page "Square triangular numbers".

I'm going to give you two ways to connect it to the Pell equation $a^2 - 2b^2 = \pm 1$ that I was talking about earlier. The first one is uses the idea that M and $M+1$ are a^2 and $2b^2$ in some order, and it can be proved that this is the only way that $\frac{M(M+1)}{2}$ can be written as a square.

$$M = 2b^2, M+1 = a^2 \implies \frac{(2b^2)(a^2)}{2} = (ab)^2, \quad a^2 - 2b^2 = 1$$

$$M = a^2, M+1 = 2b^2 \implies \frac{(a^2)(2b^2)}{2} = (ab)^2, \quad a^2 - 2b^2 = -1.$$

An alternative direct algebraic approach involves multiplication by 8 and addition by one

$$\frac{M(M+1)}{2} = n^2 \iff 8\frac{M(M+1)}{2} + 1 = 8n^2 + 1$$
$$\iff 4M^2 + 4M + 1 = 8n^2 + 1 \iff (2M+1)^2 - 2(2n)^2 = 1.$$

This takes only the solutions to $a^2 - 2b^2 = 1$ in which a is odd and b is even and then $M = \frac{a-1}{2}$ and $n = \frac{b}{2}$.

Even though this seems like an entirely different approach to the first one, the two are deeply connected.

Let me bring back the solutions to $a^2 - 2b^2 = \pm 1$ from the continued fraction:

$$\frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}, \quad \frac{41}{29}, \quad \frac{99}{70}, \quad \frac{239}{169}, \quad \frac{577}{408}, \quad \ldots.$$

Reading these off and remembering that $\{M, M+1\} = \{a^2, 2b^2\}$ in some order and $n = ab$:

$$a = 3, b = 2; \quad a^2 = 9,\ 2b^2 = 8,\ ab = 6, \quad 6^2 = \frac{8 \cdot 9}{2};$$
$$a = 7, b = 5; \quad a^2 = 49,\ 2b^2 = 50,\ ab = 35, \quad 35^2 = \frac{49 \cdot 50}{2};$$
$$a = 17, b = 12;\ a^2 = 289,\ 2b^2 = 288,\ ab = 204,\ 204^2 = \frac{288 \cdot 289}{2}.$$

and $n = ab = 204, M = 288$ is the answer to the *Strand* problem, but you can keep going on and give as many solutions as you might like. The next larger one is $n = 41 \cdot 29 = 1189$, $M = 1681$.

If you take the second approach and look for a solution to $a^2 - 2b^2 = 1$ in which $a = 2M + 1$ is odd and $b = 2n$ is even, you get the same answer, but in a different way:

$$\begin{gathered} a = 2M + 1 = 17,\ \ b = 2n = 12 \\ \implies M = 8,\ \ n = 6,\ \ 6^2 = \frac{8 \cdot 9}{2}; \\ a = 2M + 1 = 99,\ \ b = 2n = 70 \\ \implies M = 49,\ \ n = 35,\ \ 35^2 = \frac{49 \cdot 50}{2}; \\ a = 2M + 1 = 577,\ \ b = 2n = 408 \\ \implies M = 288,\ \ n = 204,\ \ 204^2 = \frac{288 \cdot 289}{2}. \end{gathered}$$

This even tells us the connections within the approximations to the continued fraction:

$$\frac{577}{408} = \frac{288 + 289}{408} = \frac{2 \cdot 12^2 + 17^2}{2 \cdot 12 \cdot 17}; \quad \frac{a_{2n}}{b_{2n}} = \frac{a_n^2 + 2b_n^2}{2a_n b_n}.$$

For those of you who know Newton's method in Calculus: if you guess $\frac{17}{12}$ as a root to the equation $x^2 - 2 = 0$, then the method will give back $\frac{577}{408}$ as a better approximation, and so with all the others.

To sum up the significance of the *Strand* problem. It's not just that Ramanujan instantly recognized the square triangular problem, it's that he instantly recalled the continued fraction which gives all the solutions. While stirring vegetables.

I have to end with a famous anecdote from G. H. Hardy:

> *He could remember the idiosyncrasies of numbers in an almost uncanny way. It was Littlewood who said that every positive integer was one of Ramanujan's personal friends. I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."*

Hardy also made this evaluation at the end of his preface to the *Collected Papers of Srinivasa Ramanujan*, published in 1927:

> *Opinions may differ as to the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on the mathematics of the future. It has not the simplicity and inevitableness of the very greatest work; it would be greater if it were less strange. One gift it has which no one can deny, profound and invincible originality. He would probably have been a greater mathematician if he had been caught and tamed a little in his youth; he would have discovered more that was new, and that, no doubt, of greater importance. On the other hand, he would have been less of a Ramanujan and more of a European professor, and the loss might have been greater than the gain.*

My final words to you are these: Mathematics is a wonderful subject. Don't you believe that it is just for young people! It has been sustaining me all my life and this is true for many, many mathematicians.

Ramanujan's brilliance has been a continuing gift to the art and science of mathematics. His unbounded originality has inspired mathematicians for the last hundred years and I hope will continue to do so for you as well.

Speaking to you from ten thousand miles away, let me end by saying:

Thank you for your attention.