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AN INEQUALITY FOR PRODUCTS OF POLYNOMIALS

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ABSTRACT. Beauzamy, Bombieri, Enflo, and Montgomery recently established an inequality for the coefficients of products of homogeneous polynomials in several variables with complex coefficients (forms). We give this inequality an alternative interpretation: let f be a form of degree m, let $f(D)$ denote the associated mth order differential operator, and define $\|f\|$ by $\|f\|^2 = f(D)\overline{f}$. Then $\|pq\| \geq \|p\|\,\|q\|$ for all forms p and q, regardless of degree or number of variables. Our principal result is that $\|pq\| = \|p\|\,\|q\|$ if and only if, after a unitary change of variables, p and q are forms in disjoint sets of variables. This is achieved via an explicit formula for $\|pq\|^2$ in terms of the coefficients of p and q.

1. INTRODUCTION

Recently, Beauzamy, Bombieri, Enflo, and Montgomery [1, Theorem 1.2] established an inequality for the coefficients of products of homogeneous polynomials with complex coefficients (forms). They defined a norm $[p]_2$ on forms (see (2.13)) and showed that if p and q are forms in n variables with degrees d and e, respectively, then

$$[pq]_2 \geq \frac{(d!e!)^{1/2}}{((d+e)!)^{1/2}}[p]_2[q]_2. \tag{1.1}$$

They remarked that (1.1) holds independently of the number of variables. It is trivial to observe that if we set

$$\|f\| = ((\deg f)!)^{1/2}[f]_2, \tag{1.2}$$

then (1.1) becomes an inequality that holds for forms independently of both the number of variables and the degree:

$$\|pq\| \geq \|p\|\,\|q\|. \tag{1.3}$$

In this paper, we shall give a less contrived definition of $\|\cdot\|$ and study the case of equality in (1.3).

In §2 we discuss the vector spaces of forms of fixed degree in a fixed number of variables. For a form f of degree d, let $f(D)$ be the associated dth order

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differential operator. We show that $\|f\|^2 = f(D)\overline{f}$, and if f and g are related by a unitary change of variables then $\|f\| = \|g\|$.

In §3 we prove the main new result of this paper. We say that two forms p and q are "unitarily disjoint" if, after a unitary (linear) change of variables, they depend on disjoint sets of variables. Equivalently, p and q are unitarily disjoint if there exists an orthogonal decomposition $\mathbb{C}^n = A \oplus B$ so that, for all x, $\nabla p(x) \in A$ and $\nabla q(x) \in B$.

Main Theorem. *Let p and q be forms. Then $\|pq\| = \|p\|\,\|q\|$ if and only if p and q are unitarily disjoint.*

Beauzamy [2] showed (using combinatorial identities) that (1.1) (and therefore (1.3) as well) is sharp for $p(x_1, x_2) = (x_1+x_2)^d$ and $q(x_1, x_2) = (x_1-x_2)^d$. The main theorem implies that the only such examples among binary forms have the shape $p(x_1, x_2) = \lambda(\alpha x_1 + \beta x_2)^d$ and $q(x_1, x_2) = \mu(\overline{\beta} x_1 - \overline{\alpha} x_2)^e$ for complex numbers λ, μ, α, and β and integers d and e. (Two of Beauzamy's students, J.-L. Frot and C. Millour (see [3]), also proved the main theorem very recently, using the multilinear approach of [1, 2]. Their proof and ours are substantially different.)

Our proof of the main theorem relies on an explicit and complicated formula for $\|pq\|^2$, inspired by a reading of [1], that leads immediately to an independent proof of (1.3). We give a combinatorial proof of this product formula in §4. Section 5 contains some implications of the formula and possible areas of future work.

2. The vector space of n-ary d-ic forms

The contents of this section are very similar to parts of [5], in which attention is restricted to real forms.

Fix $n \geq 2$ and $d \geq 1$, and let $\mathscr{F}_{n,d}$ denote the vector space of homogeneous polynomials $p(x_1, \ldots, x_n)$ of degree d. We write the elements of $\mathscr{F}_{n,d}$ as follows. Let $\mathscr{I}(n, d)$ denote the index set for the monomials:

$$(2.1) \qquad \mathscr{I}(n, d) = \left\{ i = (i_1, \ldots, i_n) : 0 \leq i_r \in \mathbb{Z},\ \sum_r i_r = d \right\}.$$

(Here and throughout the paper, a sum or product on r shall run from 1 to n.) For $i \in \mathscr{I}(n, d)$ and an n-tuple x of indeterminates, we write x^i for $\prod_r x_r^{i_r}$, and for $i \in \mathscr{I}(n, d)$ we define the multinomial coefficient

$$(2.2) \qquad c(i) = d!/\prod_r i_r!.$$

A typical element p of $\mathscr{F}_{n,d}$ is written

$$(2.3) \qquad p(x) = \sum_{i \in \mathscr{I}(d)} c(i) a(p; i) x^i,$$

where $a(p; i) \in \mathbb{C}$ and $\mathscr{I}(d)$ is short for $\mathscr{I}(n, d)$ in sums, here and throughout.

We also define a complex inner product on $\mathscr{F}_{n,d}$, whose roots go back to nineteenth century projective geometry, and which has been used sporadically in the twentieth century (see, e.g., [4–6]). For $p, q \in \mathcal{F}_{n,d}$, let

$$[p, q] = \sum_{i \in \mathcal{I}(d)} c(i) a(p; i) \overline{a(q; i)}. \tag{2.4}$$

Among several interesting properties of $[\cdot, \cdot]$ is the way in which it is a reproducing kernel. For $\alpha \in \mathbb{C}^n$, define $(\alpha \cdot)^d \in \mathcal{F}_{n,d}$ by

$$(\alpha\cdot)^d(x) = (\alpha \cdot x)^d = \left(\sum_r \alpha_r x_r\right)^d = \sum_{i \in \mathcal{I}(d)} c(i) \alpha^i x^i, \tag{2.5}$$

the last identity being the multinomial theorem. Then for $p \in \mathcal{F}_{n,d}$,

$$[p, (\alpha\cdot)^d] = \sum_{i \in \mathcal{I}(d)} c(i) a(p; i) \overline{\alpha}^i = p(\overline{\alpha}). \tag{2.6}$$

Every $p \in \mathcal{F}_{n,d}$ defines a dth order differential operator. Let $D^i = (\partial/\partial x_1)^{i_1} \cdots (\partial/\partial x_n)^{i_n}$ for $i \in \mathcal{I}(n, d)$, and define

$$p(D) = \sum_{i \in \mathcal{I}(d)} c(i) a(p; i) D^i. \tag{2.7}$$

Then for $q \in \mathcal{F}_{n,d}$,

$$p(D)\overline{q} = \sum\sum_{i, i' \in \mathcal{I}(d)} c(i) c(i') a(p; i) \overline{a(q; i')} D^i x^{i'}. \tag{2.8}$$

If $i_r > i'_r$ for any index r, then $D^i x^{i'} = 0$; since $\sum_r i_r = \sum_r i'_r = d$, it follows that $D^i x^{i'} \neq 0$ if and only if $i = i'$. Since $D^i x^i = \prod_r (i_r)! = d!/c(i)$, we have

$$p(D)\overline{q} = d! \sum_{i \in \mathcal{I}(d)} c(i) a(p; i) \overline{a(q; i)} = d![p, q]. \tag{2.9}$$

The inner product has a useful contravariant property, which is proved using a familiar folklore result. Suppose $p \in \mathcal{F}_{n,d}$ and M is an $n \times n$ complex matrix. View x as a column vector and define $p \circ M$ by

$$(p \circ M)(x) = p(Mx), \tag{2.10}$$

so $(p \circ M_1) \circ M_2 = p \circ (M_2 M_1)$. If $p = (\alpha\cdot)^d$, then $p \circ M = (\alpha M \cdot)^d$, where $\alpha \in \mathbb{C}^n$ is viewed as a row vector.

Lemma 2.11 (see [5, Proposition 2.7]). *For all n and d,*

$$\mathcal{F}_{n,d} = \operatorname{span}\{(\alpha\cdot)^d : \alpha \in \mathbb{C}^n\}.$$

Proof. If $p \in \operatorname{span}\{(\alpha\cdot)^d : \alpha \in \mathbb{C}^n\}^\perp$ then, by (2.6), $p(\overline{\alpha}) = 0$ for all $\alpha \in \mathbb{C}^n$, hence $p = 0$. □

It follows from this lemma that there exists a "basic set of nodes": $\{\alpha_k : 1 \leq k \leq |\mathcal{I}(n, d)|\} \subset \mathbb{C}^n$ so that $\{(\alpha_k \cdot)^d\}$ spans $\mathcal{F}_{n,d}$. Although we shall not need it here, $\mathcal{I}(n, d)$ itself is a basic set of nodes (see [5, Proposition 2.11]).

Theorem 2.12 (see [5, Theorem 2.15]). *Suppose* p, $q \in \mathscr{F}_{n,d}$, *and* M *is a complex* $n \times n$ *matrix with adjoint* M^*. *Then* $[p \circ M, q] = [p, q \circ M^*]$. *In particular, if* M *is unitary then* $[p \circ M, q \circ M] = [p, (q \circ M) \circ M^*] = [p, q \circ M^*M] = [p, q]$.

Proof. By linearity and the last lemma, it suffices to prove that $[p \circ M, q] = [p, q\circ M^*]$ when p and q are dth powers. By (2.6), $[(\alpha\cdot)^d, (\beta\cdot)^d] = (\sum \alpha_r \overline{\beta_r})^d = (\alpha, \beta)^d$, where $(\cdot, \cdot)$ is the usual $\mathbb{C}^n$ inner product. Thus, if $p = (\alpha\cdot)^d$ and $q = (\beta\cdot)^d$, then $[p \circ M, q] = [(\alpha M\cdot)^d, (\beta\cdot)^d] = (\alpha M, \beta)^d = (\alpha, \beta M^*)^d = [p, q \circ M^*]$. □

This theorem can also be proved by establishing it first for $p = x^i$ and $q = x^{i'}$; the resulting calculation is somewhat more involved.

In [1, Introduction] the norms $[p]_r$, $1 \le r < \infty$, are defined as

$$(2.13) \qquad [p]_r^r = \sum_{i \in \mathscr{I}(d)} c(i)^{1-r}|c(i)a(p; i)|^r = \sum_{i \in \mathscr{I}(d)} c(i)|a(p; i)|^r.$$

We see from (2.4) and (2.13) that $[p]_2^2 = [p, p]$, and from (1.2) and (2.9) that $\|p\|^2 = D(p)\overline{p} = d![p]_2^2$.

Corollary 2.14. *If* $p \in \mathscr{F}_{n,d}$ *and* M *is unitary, then* $\|p\| = \|p \circ M\|$.

Proof. We have $\|p\|^2 = d![p, p] = d![p \circ M, p \circ M] = \|p \circ M\|^2$ by Theorem 2.12. □

We conclude this section with some remarks about $p \circ M$. Suppose $M = [m_{ij}]$ and let $x_j' = \sum_r m_{jr}x_r$. Then, by (2.10),

$$(2.15) \qquad (p \circ M)(x_1, \dots, x_n) = p(x_1', \dots, x_n'),$$

which is not, strictly speaking, a change of variables for p. However, if M is unitary and $x_j^* = \sum_r \overline{m}_{rj}x_r$, then (2.15) inverts to

$$(2.16) \qquad p(x_1, \dots, x_n) = (p \circ M)(x_1^*, \dots, x_n^*).$$

3. The proof of the main theorem

We say that $p \in \mathscr{F}_{n,d}$ and $q \in \mathscr{F}_{n,e}$ are *unitarily disjoint* if there exists a unitary matrix M and $1 \le t \le n$ so that $p \circ M$ is a form in $x_1, \dots, x_t$ and $q \circ M$ is a form in $x_{t+1}, \dots, x_n$. We see by (2.16) that this is equivalent to p and q being forms in $\{x_1^*, \dots, x_t^*\}$ and $\{x_{t+1}^*, \dots, x_n^*\}$, respectively.

Lemma 3.1. *The forms* p *and* q *are unitarily disjoint if and only if there is an orthogonal decomposition* $\mathbb{C}^n = A \oplus B$ *such that, for all* $x \in \mathbb{C}^n$, $\nabla p(x) \in A$ *and* $\nabla q(x) \in B$.

Proof. Suppose that p and q are unitarily disjoint and M is the unitary matrix of the definition. For $1 \le r \le n$, let $D_r = \partial/\partial x_r$, so that $D_r(q \circ M) = 0$ for $1 \le r \le t$ and $D_r(p \circ M) = 0$ for $t + 1 \le r \le n$ by hypothesis. Let m_r denote the rth column of M. Then (2.15) implies that $D_r(f \circ M) = (m_r, \overline{\nabla f})$. Thus the desired decomposition of $\mathbb{C}^n$ occurs with $A = \operatorname{span}\{\overline{m}_1, \dots, \overline{m}_t\}$ and $B = \operatorname{span}\{\overline{m}_{t+1}, \dots \overline{m}_n\}$. Conversely, if such a decomposition exists, we may construct M, as above, out of orthonormal bases for A and B. □

One direction of the proof of the main theorem is now easily established.

Theorem 3.2. *Suppose $p \in \mathscr{F}_{n,d}$ and $q \in \mathscr{F}_{n,e}$ and p and q are unitarily disjoint. Then $\|pq\| = \|p\|\,\|q\|$.*

Proof. By Corollary 2.14, we may assume without loss of generality that p and q already involve disjoint sets of variables. To emphasize this point, we write $p = p(x)$, $q = q(y)$ and $p(D) = p(D_x)$, $q(D) = q(D_y)$. We have

$$pq(D) = \sum_{i \in \mathscr{I}(d)} \sum_{j \in \mathscr{I}(e)} c(i)c(j)a(p\,;\,i)a(q\,;\,j)D_x^i D_y^j, \tag{3.3}$$

$$\overline{pq} = \sum_{i' \in \mathscr{I}(d)} \sum_{j' \in \mathscr{I}(e)} c(i')c(j')\overline{a(p\,;\,i')a(q\,;\,j')}x^{i'}y^{j'}. \tag{3.4}$$

Since the variables are disjoint, $D_x^i D_y^j x^{i'} y^{j'} = \{D_x^i x^{i'}\}\{D_y^j y^{j'}\}$, which vanishes unless $i = i'$ and $j = j'$, when it equals $\{\prod_r (i_r)!\}\{\prod_r (j_r)!\}$. Thus

$$\begin{aligned} \|pq\|^2 &= pq(D)\overline{pq} \\ &= \sum_{i \in \mathscr{I}(d)} \sum_{j \in \mathscr{I}(e)} \prod_r (i_r)! \prod_r (j_r)! c(i)^2 c(j)^2 |a(p\,;\,i)|^2 |a(q\,;\,j)|^2 \\ &= \left\{ \sum_{i \in \mathscr{I}(d)} d!c(i)|a(p\,;\,i)|^2 \right\} \left\{ \sum_{j \in \mathscr{I}(e)} e!c(j)|a(q\,;\,j)|^2 \right\} \\ &= \|p\|^2 \|q\|^2. \quad \square \end{aligned} \tag{3.5}$$

The proof of the converse is based on Formula 3.7, which gives $\|pq\|^2$ in terms of p and q; we also obtain an independent proof of (1.3). We defer the proof of Formula 3.7 to the next section.

We shall need some notation. Suppose $p \in \mathscr{F}_{n,d}$ and $q \in \mathscr{F}_{n,e}$, and suppose $0 \le k \le \min(d, e)$. For $\alpha \in \mathscr{I}(n, d-k)$ and $\beta \in \mathscr{I}(n, e-k)$, define

$$A_k(p, q\,;\,\alpha, \beta) = \sum_{\gamma \in \mathscr{I}(k)} c(\gamma)a(p\,;\,\alpha + \gamma)\overline{a(q\,;\,\beta + \gamma)}. \tag{3.6}$$

We make some observations about A_k. If $k = 0$, then $\gamma = (0, \dots, 0)$ is the only term in the sum and $c(\gamma) = 1$, so $A_0(p, q\,;\,\alpha, \beta) = a(p\,;\,\alpha)\overline{a(q\,;\,\beta)}$. If $d = e = k$, then α and β must be $(0, \dots, 0)$ and $A_k(p, q\,;\,0, 0) = [p, q]$. We shall see in Lemma 3.13 that $A_1(p, q\,;\,\alpha, \beta)$ is closely related to ∇p and ∇q.

Formula 3.7. *Suppose $p \in \mathscr{F}_{n,d}$ and $q \in \mathscr{F}_{n,e}$. Then*

$$\|pq\|^2 = d!e! \sum_k \binom{d}{k}\binom{e}{k} \sum_{\alpha \in \mathscr{I}(d-k)} \sum_{\beta \in \mathscr{I}(e-k)} c(\alpha)c(\beta)|A_k(p, q\,;\,\alpha, \beta)|^2, \tag{3.8}$$

where the sum is taken for $k \ge 0$ and $A_k(p\,;\,q\,;\,\alpha, \beta)$ is given by (3.6).

Theorem 3.9. *If $p \in \mathscr{F}_{n,d}$ and $q \in \mathscr{F}_{n,e}$, then $\|pq\| \ge \|p\|\,\|q\|$, with equality if and only if $A_k(p, q\,;\,\alpha, \beta) = 0$ for all $\alpha \in \mathscr{I}(n, d-k)$ and $\beta \in \mathscr{I}(n, e-k)$ with $k \ge 1$.*

Proof. By taking the terms for $k = 0$ in (3.8), we obtain the inequality

$$(3.10)\quad \begin{aligned} \|pq\|^2 &\geq d!e! \sum_{\alpha\in\mathcal{I}(d)} \sum_{\beta\in\mathcal{I}(e)} c(\alpha)c(\beta)|A_0(p, q; \alpha, \beta)|^2 \\ &= d!e! \sum_{\alpha\in\mathcal{I}(d)} \sum_{\beta\in\mathcal{I}(e)} c(\alpha)c(\beta)|a(p; \alpha)|^2|a(q; \beta)|^2 = \|p\|^2\|q\|^2. \end{aligned}$$

Since each $c(i)$ is positive, the remark on equality is immediate. □

We turn our attention to A_1. Suppose $f \in \mathcal{F}_{n,t}$. Let $e_1, \dots, e_n$ denote the usual unit vectors, and for $\mu \in \mathcal{I}(n, t-1)$, let

$$(3.11)\quad \Lambda_{f;\mu} = (a(f; \mu + e_1), \dots, a(f; \mu + e_n)) \in \mathbb{C}^n.$$

Since $c(\gamma) = 1$ for $\gamma \in \mathcal{I}(n, 1) = \{e_j\}$, we may rewrite A_1 (cf. (3.6)) more succinctly in terms of the $\mathbb{C}^n$ inner product:

$$(3.12)\quad A_1(p, q; \alpha, \beta) = (\Lambda_{p;\alpha}, \Lambda_{q;\beta}).$$

Lemma 3.13. *If $f \in \mathcal{F}_{n,t}$, then*

$$\nabla f = t \sum_{\mu\in\mathcal{I}(t-1)} c(\mu)\Lambda_{f,\mu} x^{\mu}.$$

Proof. It suffices by symmetry to compute $\partial f/\partial x_1$ for

$$f(x) = \sum_{l\in\mathcal{I}(t)} c(l)a(f; l)x^l :$$

$$(3.14)\quad \frac{\partial f}{\partial x_1} = \sum_{l\in\mathcal{I}(t)} l_1 c(l)a(f; l)x^{l-e_1}.$$

We rewrite this sum by letting $\mu = l - e_1 \in \mathcal{I}(n, t-1)$. Note that if $l - e_1 \notin \mathcal{I}(n, t-1)$ then $l_1 = 0$ and the term does not occur in the sum. Note also that $l_1 c(l) = (\mu_1 + 1)c(\mu + e_1) = tc(\mu)$. Thus, (3.14) becomes

$$(3.15)\quad \frac{\partial f}{\partial x_1} = \sum_{\mu\in\mathcal{I}(t-1)} tc(\mu)a(f; \mu + e_1)x^{\mu}. \quad \square$$

Theorem 3.16. *If $p \in \mathcal{F}_{n,d}$ and $q \in \mathcal{F}_{n,e}$ and $A_1(p, q; \alpha, \beta) = 0$ for all $\alpha \in \mathcal{I}(n, d-1)$ and $\beta \in \mathcal{I}(n, e-1)$, then p and q are unitarily disjoint.*

Proof. For $f \in \mathcal{F}_{n,t}$, let $\Lambda_f = \operatorname{span}\{\Lambda_{f;\mu}: \mu \in \mathcal{I}(t-1)\}$. By hypothesis, (3.11), and (3.12), $(\Lambda_{p;\alpha}, \Lambda_{q;\beta}) = 0$ for all (α, β), and we may extend, if necessary, $\Lambda_p \subseteq A$ and $\Lambda_q \subseteq B$ into an orthogonal decomposition of $\mathbb{C}^n$. It follows by Lemma 3.1 that p and q are unitarily disjoint. □

Proof of the main theorem. Combine Theorems 3.2, 3.9, and 3.16. □

4. The proof of Formula 3.7

It seems likely that any proofs of this formula will be combinatorial, not conceptual. We discovered it by a careful counting of the terms that arise after all the shuffles in Theorem 1.B.2 of [1] but give a new proof here.

We compute $\|pq\|^2 = pq(D)\overline{pq}$ for $p \in \mathscr{F}_{n,d}$ and $q \in \mathscr{F}_{n,e}$. Since

$$p(x)q(x) = \sum_{i\in\mathscr{I}(d)} \sum_{j\in\mathscr{I}(e)} c(i)c(j)a(p\,;\,i)a(q\,;\,j)x^{i+j}, \tag{4.1}$$

by definition, we have

$$\|pq\|^2 = \sum\sum_{i,i'\in\mathscr{I}(d)} \sum\sum_{j,j'\in\mathscr{I}(e)} C(i\,,\,i'\,,\,j\,,\,j')a(p\,;\,i)a(q\,;\,j)\overline{a(p\,;\,i')a(q\,;\,j')}, \tag{4.2}$$

where

$$C(i\,,\,i'\,,\,j\,,\,j') = c(i)c(i')c(j)c(j')D^{i+j}x^{i'+j'}. \tag{4.3}$$

As in (2.8), $C(i\,,\,j\,,\,i'\,,\,j') = 0$ if $i+j \neq i'+j'$ and

$$C(i\,,\,j\,,\,i'\,,\,j') = c(i)c(i')c(j)c(j')\prod_r (l_r!), \qquad \text{if } i+j = i'+j' = l. \tag{4.4}$$

Observe that for $\alpha \in \mathscr{I}(d-k)$ and $\beta \in \mathscr{I}(e-k)$,

$$\begin{aligned} &|A_k(p\,,\,q\,;\,\alpha\,,\,\beta)|^2 \\ &= \left|\sum_{\gamma\in\mathscr{I}(k)} c(\gamma)a(p\,;\,\alpha+\gamma)\overline{a(q\,;\,\beta+\gamma)}\right|^2 \\ &= \sum\sum_{\gamma,\delta\in\mathscr{I}(k)} c(\gamma)c(\delta)a(p\,;\,\alpha+\gamma)\overline{a(q\,;\,\beta+\gamma)a(p\,;\,\alpha+\delta)}a(q\,;\,\beta+\delta). \end{aligned} \tag{4.5}$$

Thus, a typical term on the right-hand side of (3.8) is

$$d!e!\binom{d}{k}\binom{e}{k}c(\alpha)c(\beta)c(\gamma)c(\delta)a(p\,;\,\alpha+\gamma)\overline{a(q\,;\,\beta+\gamma)a(p\,;\,\alpha+\delta)}a(q\,;\,\beta+\delta). \tag{4.6}$$

A comparison with (4.2) shows that upon writing

$$i = \alpha+\gamma, \qquad j = \beta+\delta, \qquad i' = \alpha+\delta, \qquad j' = \beta+\gamma, \tag{4.7}$$

every term in (4.6) satisfies $i+j = i'+j' = l$ (say). Let

$$D(i\,,\,j\,,\,i'\,,\,j') := \sum d!e!\binom{d}{k}\binom{e}{k}c(\alpha)c(\beta)c(\gamma)c(\delta), \tag{4.8}$$

where the sum is taken over all $(\alpha\,,\,\beta\,,\,\gamma\,,\,\delta)$ satisfying (4.7). Comparing (4.4) and (4.8), we see that Formula 3.7 is established if we can show that $C(i\,,\,j\,,\,i'\,,\,j') = D(i\,,\,j\,,\,i'\,,\,j')$ whenever $i+j = i'+j'$. We do this combinatorially.

Let $s\,,\,t\,,\,u\,,\,v$ each denote n-tuples, and define

$$\begin{aligned} F(s\,,\,t\,,\,u\,,\,v) &= \left(\sum_r (s_r+t_r)(u_r+v_r)\right)^{d+e} \\ &= \left(\sum_r s_r u_r + \sum_r s_r v_r + \sum_r t_r u_r + \sum_r t_r v_r\right)^{d+e} \\ &= \sum\sum_{i,i'\in\mathscr{I}(d)} \sum\sum_{j,j'\in\mathscr{I}(e)} E(i\,,\,i'\,,\,j\,,\,j')s^i t^j u^{i'} v^{j'}. \end{aligned} \tag{4.9}$$

We compute $E(i, i', j, j')$ by two applications of the multinomial theorem.

First,

$$F(s, t, u, v) = \sum_{\lambda \in \mathscr{I}(d+e)} c(\lambda) \prod_r \{(s_r + t_r)(u_r + v_r)\}^{\lambda_r}. \tag{4.10}$$

The term in (4.10) that contains $s^i t^j u^{i'} v^{j'}$ occurs for $\lambda = i + j = i' + j' = l$. Thus, $E(i, i', j, j')$ is the coefficient of $s^i t^j u^{i'} v^{j'}$ in

$$c(l) \prod_r (s_r + t_r)^{l_r} (u_r + v_r)^{l_r}; \tag{4.11}$$

namely,

$$\begin{aligned} \frac{(d+e)!}{\prod_r l_r!} \frac{\prod_r l_r!}{\prod_r i_r! j_r!} \frac{\prod_r l_r!}{\prod_r i'_r! j'_r!} &= \frac{(d+e)! \prod_r l_r!}{(d!)^2 (e!)^2} c(i) c(j) c(i') c(j') \\ &= (d+e)! (d!)^{-2} (e!)^{-2} C(i, i', j, j'). \end{aligned} \tag{4.12}$$

We may also expand $F(s, t, u, v)$ as a quadrinomial:

$$\begin{aligned} F(s, t, u, v) = \sum_{\sigma \in \mathscr{I}(d+e, 4)} c(\sigma) \left(\sum_r s_r u_r\right)^{\sigma_1} \left(\sum_r s_r v_r\right)^{\sigma_2} \\ \times \left(\sum_r t_r u_r\right)^{\sigma_3} \left(\sum_r t_r v_r\right)^{\sigma_4}. \end{aligned} \tag{4.13}$$

The term $s^i t^j u^{i'} v^{j'}$ occurs in (4.13) for those σ with $\sigma_1 + \sigma_2 = \sigma_1 + \sigma_3 = d$ and $\sigma_2 + \sigma_4 = \sigma_3 + \sigma_4 = e$; thus, $\sigma = (d-k, k, k, e-k)$ for $0 \le k \le \min(d, e)$. Therefore the coefficient of $s^i t^j u^{i'} v^{j'}$ in (4.13) is

$$\sum_{\alpha, \beta, \gamma, \delta} c(\sigma) c(\alpha) c(\gamma) c(\delta) c(\beta), \tag{4.14}$$

where the sum is over those $\alpha \in \mathscr{I}(n, d-k)$, $\beta \in \mathscr{I}(n, e-k)$, $\gamma \in \mathscr{I}(n, k)$, and $\delta \in \mathscr{I}(n, k)$, any k, satisfying (4.7). Since $c(\sigma) = \binom{d}{k}\binom{e}{k}\binom{d+e}{e}$, (4.8) implies that

$$E(i, j, i', j') = (d+e)! (d!)^{-2} (e!)^{-2} C(i, i', j, j'). \tag{4.15}$$

Comparison with (4.12) shows that $C(i, i', j, j') = D(i, j, i', j')$, and we are done.

5. Final remarks and open questions

For a fixed form f, consider the ratio $L_f(p) = \|fp\|^2 / (\|f\|^2 \|p\|^2)$ as p ranges over $\mathscr{F}_{n,d}$. Since $L_f(\lambda p) = L_f(p)$, we may restrict our attention to p with $\|p\| = 1$, hence the following quantities are well defined:

$$M_{n,d}(f) = \max\{L_f(p) : p \in \mathscr{F}_{n,d}\}, \tag{5.1)(i}$$

$$m_{n,d}(f) = \min\{L_f(p) : p \in \mathscr{F}_{n,d}\}. \tag{5.1)(ii}$$

If n is larger than the dimension of the span of ∇f, then $m_{n,d}(f) = 1$, and $L_f(p) = 1$ only if p and f involve unitarily disjoint variables. In any case,

$$M_{n,d}(f) \|p\|^2 \|q\|^2 \ge \|pq\|^2 \ge m_{n,d}(f) \|p\|^2 \|q\|^2. \tag{5.2}$$

The computation of $M_{n,d}(f)$ and $m_{n,d}(f)$ appear to be complicated in general. We discuss some simpler special cases. Suppose $f(x) = x^l \in \mathscr{F}_{n,t}$ where $l_1 \geq l_2 \geq \cdots \geq l_n$. If $p(x) = \sum_{i \in \mathscr{I}(d)} c(i)a(p; i)x^i \in \mathscr{F}_{n,d}$ then

$$
\begin{aligned}
L_f(p) &= \frac{\| \sum_{i \in \mathscr{I}(d)} c(i)a(p; i)x^{i+l} \|^2}{\|x^l\|^2 \| \sum_{i \in \mathscr{I}(d)} c(i)a(p; i)x^i \|^2} \\
&= \frac{\sum_{i \in \mathscr{I}(d)} \{\prod_r (i_r + l_r)!\} c(i)^2 |a(p; i)|^2}{\sum_{i \in \mathscr{I}(d)} \{\prod_r l_r!\} \{\prod_r (i_r!)\} c(i)^2 |a(p; i)|^2}.
\end{aligned} \tag{5.3}
$$

It is clear from (5.3) that $M_{n,d}(f)$ and $m_{n,d}(f)$ are, respectively, the maximum and minimum values of the quantity

$$
R(i, l) = \prod_{r=1}^{n} \frac{(i_r + l_r)!}{i_r! l_r!} = \prod_{r=1}^{n} \left\{ \prod_{k=1}^{l_r} \left(1 + \frac{l_r}{k} \right) \right\}, \tag{5.4}
$$

as i ranges over $\mathscr{I}(n, d)$. It is not hard to prove that $R(i, l)$ achieves its minimum $\binom{l_n+d}{d}$ at $i = (0, 0, \ldots, d)$, for $p(x) = x_n^d$. In particular, if $l_n = 0$ then $m_{n,d}(f) = 1$, as one would expect, since f does not involve x_n. The behavior of the maximum of $R(i, l)$ is more delicate; it is achieved by taking the d largest numbers of the form $1 + l_r/k$. Roughly speaking, this means taking i as close to $(d/m)l$ as is possible in $\mathscr{I}(n, d)$. We obtain the asymptotic estimate $M_{n,d}(f) \approx C \cdot (ed/m + o(1))^d$ by Stirling's formula. (Here, $e \approx 2.718$ is Euler's constant, not a degree.)

If $f \in \mathscr{F}_{2,2}$ then after a unitary change of variables, we may assume that $f(x_1, x_2) = \lambda\{(\cos\alpha)x_1^2 - (\sin\alpha)x_2^2\} := \lambda f_\alpha(x)$. We can show in this case that $M_{2,2}(f_\alpha)$ and $m_{2,2}(f_\alpha)$ are the two roots of the quadratic

$$
\Phi_\alpha(t) = t^2 - 7t + 6(1 + \sin^2 2\alpha). \tag{5.5}
$$

This can be checked against the previous paragraph. If $l = (2, 0)$ then $x^l = x_1^2 = f_0$, and the roots of Φ_0 are 1 and 6. It is easy to check that $R(i, l)$ equals 1 for $i = (0, 2)$ and 6 for $i = (2, 0)$. If $l = (1, 1)$ then $x^l = x_1x_2$, which is unitarily equivalent to $f_{\pi/4}$; the roots of $\Phi_{\pi/4}$ are 3 and 4. We see from (5.4) that $R(i, l) = 3$ for $i = (2, 0)$ or $(0, 2)$ and $R(l, l) = 4$. One might expect that, just as $L_f(p) = m_{n,d}(f)$ when f and p involve disjoint variables, then $L_f(p) = M_{n,d}(f)$ when f and p use variables in as "similar" a way as possible. The evidence given above strongly suggests that p should be as close to a power of f as possible; in particular, if $m = d$ then p should be f. This evidence is misleading: we can show that $L_{f_\alpha}(f_\alpha) = M_{2,2}(f_\alpha)$ only when $\alpha = k\pi/4$.

This discussion raises some obvious questions. What are the ranges of $M_{n,d}(f)$ and $m_{n,d}(f)$ as f runs over $\mathscr{F}_{n,e}$? Clearly, the minimum value of $m_{n,d}(f) = 1$; it is not hard to show that the maximum of $M_{n,d}(f)$ is $\binom{d+e}{e}$, see (5.10) below. The range of $M_{n,d}(f)/m_{n,d}(f)$ is also unknown.

We close this paper with two applications of the formula. Suppose

$$
p(x, y) = \prod_{k=1}^{d} \{(\cos\alpha_k)x - (\sin\alpha_k)y\} \in \mathscr{F}_{2,d} \tag{5.6}
$$

is a real binary form. Then a natural generalization of Leibniz's rule to the dth derivative of a product of d terms, whose exact statement and proof we omit, gives a closed form for $\|p\|^2$:

$$(5.7) \qquad \|p\|^2 = p(D)p = \sum_\sigma \prod_{k=1}^d \cos(\alpha_k - \alpha_{\sigma(k)}),$$

where the sum in (5.7) is taken over all permutations σ of $\{1, \dots, d\}$. In particular, the sum in (5.7) is always positive.

Finally, it is easy to compute $\|(\alpha\cdot)^d\|^2$ from (2.6) and (2.9):

$$(5.8) \qquad \|(\alpha\cdot)^d\|^2 = d![(\alpha\cdot)^d, (\alpha\cdot)^d] = d!(\alpha, \alpha)^d.$$

This begs the natural question. If $p = (\alpha\cdot)^d$ and $q = (\beta\cdot)^e$, how does $\|pq\|$ depend on α and β? There is a satisfying geometric answer.

Corollary 5.9. *If* $p = (\zeta\cdot)^d$, $q = (\xi\cdot)^e$, *and* $\tau = (\zeta, \xi)/(\|\zeta\| \cdot \|\xi\|)$, *then*

$$(5.10) \qquad \|pq\|^2 = \|p\|^2\|q\|^2 \left(\sum_k \binom{d}{k}\binom{e}{k} |\tau|^{2k} \right).$$

Proof. Since $a(p; i) = \zeta^i$ and $a(q; j) = \xi^j$, the multinomial theorem implies

$$(5.11) \qquad A_k(p, q; \alpha, \beta) = \sum_{\gamma \in \mathcal{I}(k)} c(\gamma) a(p; \alpha + \gamma)\overline{a(q; \beta + \gamma)} = \zeta^\alpha \overline{\xi}^\beta (\zeta, \xi)^k,$$

and so by Formula 3.7,

$$(5.12) \qquad \|pq\|^2 = d!e! \sum_k \binom{d}{k}\binom{e}{k} |(\zeta, \xi)^k|^2 \sum_{\alpha \in \mathcal{I}(d-k)} \sum_{\beta \in \mathcal{I}(e-k)} c(\alpha)c(\beta)|\zeta^\alpha \overline{\xi}^\beta|^2.$$

Another application of the multinomial theorem shows that the inner sum in (5.12) is $|\zeta|^{2(d-k)}|\overline{\xi}|^{2(e-k)} = (\zeta, \zeta)^{d-k}(\xi, \xi)^{e-k}$. Therefore,

$$(5.13) \qquad \|pq\|^2 = d!e! \sum_k \binom{d}{k}\binom{e}{k} |(\zeta, \xi)|^{2k} (\zeta, \zeta)^{d-k} (\xi, \xi)^{e-k}.$$

An appeal to (5.8) completes the proof. □

This corollary can also be proved by restricting without loss of generality to $(\zeta, \zeta) = (\xi, \xi) = 1$, applying Theorem 2.12 with M chosen so that $p \circ M = x_1^d$, and then using the definition of the inner product.

References

1. B. Beauzamy, E. Bombieri, P. Enflo, and H. L. Montgomery, *Products of polynomials in many variables*, J. Number Theory **36** (1990), 219–245.
2. B. Beauzamy, *Products of polynomials and a priori estimates for coefficients in polynomial decompositions: a sharp result*, J. Symbolic Comput. (to appear).
3. B. Beauzamy, J.-L. Frot, and C. Millour, *Massively parallel computations on many variable polynomials.* II (in preparation).
4. S. Helgason, *Groups and geometric analysis*, Academic Press, New York, 1984.

5. B. Reznick, *Sums of even powers of real linear forms*, Mem. Amer. Math. Soc., vol. 196, no. 463, Amer. Math. Soc., Providence, RI, 1992.

6. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, NJ, 1971.

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