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Real Algebraic Geometry with a View Toward Hyperbolic Programming and Free Probability

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Abstract. Invariants of topological spaces of dimension three play a major role in many areas, in particular ...

Introduction by the Organizers

The workshop *Invariants of topological spaces of dimension three*, organised by Max Muster (München) and Bill E. Xample (New York) was well attended with over 30 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds ...

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Workshop: Real Algebraic Geometry with a View Toward Hyperbolic Programming and Free Probability

Table of Contents

Abstracts

Lattice points in dilated simplices and sums of squares

Bruce Reznick

This talk is, in some sense, a remake of a talk I gave here in 1987, but was given with improved notation, better jokes and, unfortunately, less hair. The main theorem was proved in [2], though one announced result was not proved. This gap was remedied by Vicki Powers and the author in the recent [1]. All proofs here, except for the claim at the very end, can be found in [2] and [1].

There are two fundamental forms which arise as examples in Hilbert's 17th problem. The first is the Motzkin form

$$M(x, y, z) := x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2.$$

which is the first known example of a psd form which is not sos. (See [3] for historical details.) On the other hand, a form of a type Hurwitz studied:

$$H(x, y, z) := x^6 + y^6 + z^6 - 3x^2y^2z^2$$
$$= \tfrac{3}{2}(x^2y - yz^2)^2 + (x^3 - xy^2)^2 + \tfrac{1}{2}(x^2y - y^3)^2 + (z^3 - y^2z)^2 + \tfrac{1}{2}(yz^2 - y^3)^2$$

is evidently a sum of squares. Both these forms arise from a monomial substitution into the arithmetic-geometric inequality (AGI), and so both are psd, but M is not sos and H is sos. Why?

The answer turns out to rely on the pattern of the lattice points within the simplex determined by the vectors of monomials in the substitution. To be specific, suppose we have the AGI

$$\lambda_1 t_1 + \cdots + \lambda_n t_n \geq t_1^{\lambda_1} \cdots t_n^{\lambda_n},$$

where $t_i \geq 0$, $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. We suppose that $\{u_1, \ldots, u_n\}$ with $u_i \in (2\mathbb{Z}_{\geq 0})^n$ and $\sum_{j=1}^n u_{ij} = 2d$. We further assume that $\mathcal{U} = cvx(\{u_1, \ldots, u_n\})$ is a simplex, and that $w \in \mathcal{U} \cap \mathbb{Z}^n$ has the (unique) barycentric representation $w = \sum \lambda_i u_i$, $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. In this way, the substitution $\{t_i = x^{u_i}\}$ into the AGI yields a psd form of degree 2d,

$$p(\mathcal{U}, w)(x) := \lambda_1 x^{u_1} + \cdots + \lambda_n x^{u_n} - x^w.$$

In the two cases above, up to a multiple of 3, we have $n = 3$, $\lambda_j = 1/3$ and $\mathcal{U}_1 = \{(4, 2, 0), (2, 4, 0), (0, 0, 6)\}$ and $\mathcal{U}_2 = \{(6, 0, 0), (0, 6, 0), (0, 0, 6)\}$

Suppose $\mathcal{U}$ is defined as above and let $S \subset \mathcal{U} \cap \mathbb{Z}^n$ be a set of lattice points containing the u_i's. Then S is *$\mathcal{U}$-mediated* if for every $y \in S$, either $y = u_i$ for some i, or there exist $z_1 \neq z_2 \in S \cap (2\mathbb{Z})^n$ so that $y = \frac{1}{2}(z_1 + z_2)$. In other words, S is $\mathcal{U}$-mediated if every point in S is either a vertex of $\mathcal{U}$ or an average of two different even points in $\mathcal{U}$. (This definition generalizes naturally to polytopes, but there are no known applications.)

The following was proved in [2]: Theorem: As defined above, $p(\mathcal{U}, w)$ is sos if and only if there is a $\mathcal{U}$-mediated set containing w. The proof given in the talk evoked the formal inverse of a matrix perturbing I_m, as well as the Coen Brothers.

The paper [2] had its day in the sun and then in the way of most papers sank to the bottom of the ocean. Then the recent interest in circuit polynomials recalled it to life: mediated sets found their inner Godzilla and resurfaced. In particular, an assertion was made in [2] and not proved: For every integer $k \geq \max\{2, n-2\}$, $k\mathcal{U} \cap \mathbb{Z}^n$ is $(k\mathcal{U})$-mediated. As a corollary, each form $p(\mathcal{U}, w)(x_1^k, \ldots, x_n^k)$ is sos.

The reason this wasn't proved in [2] is that the speaker had optimistically generalized, and erroneously conjectured that such a property holds for every psd form f; in particular, this would imply that f is a sum of squares of forms in the variables $x_j^{1/k}$ for sufficiently large k.

In [1], Vicki and I give the proof of the assertion (corrected from the unpublished proof of thirty years earlier), as well as showing that this conjecture is false for the Horn form $F(x_1, \ldots, x_5)$. In the talk, a proof of a weaker version of the dilation theorem was presented: $k \geq n-1$.

What's new in the talk was the observation, based on evidence, that for the full psd-not-sos example M_n (note that $M_3 = M$) given by Motzkin in 1967:

$$M_n(x_1, \ldots, x_n) :=$$
$$x_1^4 x_2^2 \cdots x_{n-1}^2 + x_1^2 x_2^4 \cdots x_{n-1}^2 + \cdots + x_1^2 x_2^2 \cdots x_{n-1}^4 + x_n^{2n} - n x_1^2 x_2^2 \cdots x_{n-1}^2 x_n^2,$$

the lower bound $k \geq \max\{2, n-2\}$ is in fact best possible.

I want to thank the organizers for the chance to speak. I also want to thank this research community for its friendliness and openness. Young mathematicians should realize that this is not automatic! This talk was given during the penultimate Oberwolfach workshop before it was overtaken by the rough coronaviral beast which is now slouching everywhere.

References

[1] V. Powers and B. Reznick, *A note on mediated simplices*, arXiv:1909.11008, submitted for publication.

[2] B. Reznick, *Forms derived from the arithmetic-geometric inequality*, Math. Ann. **283** (1989), 431-464.

[3] B. Reznick, *Some concrete aspects of Hilbert's 17th problem*, Cont. Math. **253** (2000), 251-272

Reporter: Patrick Michalski