Every math major should know this crazy theorem

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That crazy theorem

I'll start with the theorem and two different proofs, then I'll talk about its star-studded history and then, briefly, how I came upon it.

Theorem

The solutions to the differential equation $((y'')^{-2/3})''' = 0$ *are precisely the non-degenerate conic sections.*

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Theorem

The solutions to the differential equation $((y'')^{-2/3})''' = 0$ are precisely the non-degenerate conic sections.

Proof One, Part One.

We see that $(y'')^{-2/3} = c_0 + c_1 x + c_2 x^2$ for suitable constants c_i, and, taking care to distinguish cases in the quadratic by color,

$$y'' = 2b_0, \quad \frac{b_0}{(x+b_1)^{3/2}}, \quad \frac{2b_0}{(x+b_1)^3}, \quad \frac{b_2\Delta}{(x^2+2b_1x+b_0)^{3/2}}$$

where in the last, $\Delta = b_0 - b_1^2 \neq 0$. We can easily solve these equations! $\square$

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In each case, the solution becomes a conic after simplification. □

Proof One, Part Three.

Conversely, if one solves for y in

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

then either we obtain one of the four indicated outcomes for y, or the conic is degenerate. In this case y is linear, and computing $(y'')^{-2/3}$ isn't a good idea anyway. $\square$

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$$((y'')^{-2/3})'' = \frac{2}{9(y'')^{8/3}}(5(y''')^2 - 3y''y''''),$$

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$$((y'')^{-2/3})''' = \frac{-2}{27(y'')^{11/3}}(9(y'')^2y''''' - 45y''y'''y'''' + 40(y''')^3).$$

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Also, if $x^2 + 2Bxy + y^2 = 1$, $|B| \neq 1$, then

$$((y'')^{-2/3})'' = 2(B^2 - 1)^{1/3}.$$

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$$\Delta = ACF + 2BDE - AE^2 - CD^2 - FB^2$$

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Here, Δ is the discriminant of the (homogenized) ternary quadratic form $Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2$, and $\Delta = 0$ if and only if this quadratic form has rank < 3 if and only if it is reducible if and only if the conic is degenerate. A Mathematica computation shows that, in this case,

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$$((y'')^{-2/3})'' = \frac{2(B^2 - AC)}{\Delta^{2/3}}.$$

We see that this invariant is positive on hyperbolas, zero on parabolas and negative on ellipses and circles.

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Proof Two, Part One.

Suppose

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$$

Since the coefficients are constants, by implicit differentiation, we have

$$A(2x) + 2B(xy' + y) + C(2yy') + 2D(1) + 2E(y') + F(0) = 0.$$

Do this four more times and then observe that we have a 6×6 linear system with non-trivial solution $(A, 2B, \ldots)$. $\square$

Proof Two, Part Two.

Here's the coefficient matrix in detail:

$$\begin{pmatrix} x^2 & xy & y^2 & x & y & 1 \\ 2x & xy' + y & 2yy' & 1 & y' & 0 \\ 2 & xy'' + 2y' & 2yy'' + 2(y')^2 & 0 & y'' & 0 \\ 0 & xy''' + 3y'' & 2yy''' + 6y'y'' & 0 & y''' & 0 \\ 0 & xy'''' + 4y''' & 2yy'''' + 8y'y''' + 6(y'')^2 & 0 & y'''' & 0 \\ 0 & xy''''' + 5y'''' & 2yy''''' + 10y'y'''' + 20y''y''' & 0 & y''''' & 0 \end{pmatrix}$$

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It has non-trivial nullvector $(A, 2B, C, 2D, 2E, F)^t$, so the determinant must be zero. Mathematica, helpfully, says this determinant is

$$4y''(9(y'')^2y''''' - 45y''y'''y'''' + 40(y''')^3)$$

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$$4y''(9(y'')^2y''''' - 45y''y'''y'''' + 40(y''')^3)$$
$$= -54(y'')^{-8/3}((y'')^{-2/3})'''.$$

$\square$

History

It is surprising to me that such a simple theorem seems to be so scantly represented in the literature. (Of course, I may simply be ignorant: please let me know of other sources!) This talk is too brief to do justice to the broader context of the work of these famous mathematicians.

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- George Boole (1815-1864) referred to Monge's theorem in 1844, in his text on differential equations, adding:

  > *But here our powers of geometrical interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms.*

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- Alain Lascoux (1944-2013) wrote about it in 2006 in a broader context, and his paper had references to Halphen, Monge and Sylvester. When I found the $((y'')^{-2/3})'''$ version, I wrote a group of friends to see if they knew it. One mentioned it to Tom Craven, who pointed me to Lascoux.

- Sylvester's work was part of his monumental (>200 pages) series of lectures on the Theory of Reciprocants, delivered at Oxford in 1886, three years after he became the Savilian Professor of Geometry there. Sylvester had trouble finding the original Monge reference, and apparently his colleagues in Paris couldn't help him. But "knowing how scrupulous and painstaking Boole was, and the least likely of all men to make a quotation at random", he persisted, and ultimately wrote

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  Sylvester also sought help at the "Universal Knowledge and Information Office", but there's no time to elaborate on that steampunk-y, but 100% not-made-up name!

How I got involved

Five points in the plane, no four on a line, determine a unique conic. I gave a talk on the following topic last August, at the SIAM Meeting on Applied Algebraic Geometry in Ft. Collins.

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Definition

A set $\mathcal{S}$ is *hyperbolesque* if the conic determined by any five points from $\mathcal{S}$ is a hyperbola.
A set $\mathcal{S}$ is *elliptesque* if the conic determined by any five points from $\mathcal{S}$ is an ellipse (or circle).

Assume $\mathcal{S}$ is a C^5 curve and take 5 points infinitesimally close to a point P: the sign of $5(y''')^2 - 3y''y''''$ at P determines the character of the conic. This immediately suggested looking at $((y'')^{-2/3})''$.

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If you are interested in more, go to my website on Monday or Tuesday [well, Friday]. I will have this talk, the Ft. Collins talk and links to the original documents mentioned above.

Finally, I have two excellent students, Katie Anders and Jennifer Lansing, who are completing their PhDs this year at the University of Illinois in combinatorial number theory, and who are at this conference.

Thank you!