EQUAL SUMS OF TWO CUBES OF QUADRATIC FORMS

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This paper is dedicated to my friend and colleague Bruce Berndt on his 80th birthday.

Abstract. We give a complete description of all solutions to the equation $f_1^3+f_2^3 = f_3^3 + f_4^3$ for quadratic forms $f_j \in \mathbb{C}[x, y]$ and show how Ramanujan's example can be extended to three equal sums of pairs of cubes. We also give a complete census in counting the number of ways a sextic $p \in \mathbb{C}[x, y]$ can be written as a sum of two cubes. The extreme example is $p(x, y) = xy(x^4 - y^4)$, which has six such representations.

1. Introduction

In 1913, Ramanujan [11], [12, p.326] (see [2, p.56], [6, p.201]) posed to the *Journal of the Indian Mathematical Society* the following question: "Shew that

$$\begin{gathered}(6x^2 - 4xy + 4y^2)^3 = \\ (3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3,\end{gathered} \tag{1.1}$$

and find other quadratic expressions satisfying similar relations." Write (1.1) as $R_1^3(x, y) = R_2^3(x, y) + R_3^3(x, y) + R_4^3(x, y)$ for short.

In 1914, Narayanan [10] replaced the integers in (1.1) with the variables ℓ, m, n, p and solved the resulting equations; namely, $m^3 + n^3 = p^3 - \ell^3 = mp^2 + n\ell^2$, over $\mathbb{R}$.

$$\begin{gathered}(\ell x^2 - nxy + ny^2)^3 = \\ (px^2 + mxy - my^2)^3 + (nx^2 - nxy + \ell y^2)^3 + (mx^2 - mxy - py^2)^3; \\ \ell = \lambda(\lambda^3 + 1), \quad m = 2\lambda^3 - 1, \quad n = \lambda(\lambda^3 - 2), \quad p = \lambda^3 + 1.\end{gathered} \tag{1.2}$$

Write (1.2) as $N_{1,\lambda}^3(x, y) = N_{2,\lambda}^3(x, y) + N_{3,\lambda}^3(x, y) + N_{4,\lambda}^3(x, y)$, and note $N_{j,2} = 3R_j$.

Equation (1.1) can be rewritten as two equal sums of two cubes in three different ways, and in two of the three ways, there is a third equal sum of two cubes. First,

$$\begin{aligned}&(4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3 \\ &= (6x^2 - 4xy + 4y^2)^3 - (3x^2 + 5xy - 5y^2)^3 \\ &= (6x^2 - 8xy + 6y^2)^3 - (3x^2 - 11xy + 3y^2)^3 \\ &= 63(x^2 + xy + y^2)(3x^2 - 3xy + y^2)(x^2 - 3xy + 3y^2).\end{aligned} \tag{1.3}$$

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We also have

$$
\begin{aligned}
&(6x^2 - 4xy + 4y^2)^3 - (5x^2 - 5xy - 3y^2)^3 \\
&= (4x^2 - 4xy + 6y^2)^3 + (3x^2 + 5xy - 5y^2)^3 \\
&= \left(\tfrac{94}{21}x^2 - \tfrac{8}{21}xy + \tfrac{94}{21}y^2\right)^3 + \left(\tfrac{23}{21}x^2 - \tfrac{199}{21}xy + \tfrac{23}{21}y^2\right)^3 \\
&= (13x^2 - 23xy + 13y^2)(7x^2 + xy + y^2)(x^2 + xy + 7y^2),
\end{aligned}
\tag{1.4}
$$

and

$$
\begin{aligned}
&(6x^2 - 4xy + 4y^2)^3 - (4x^2 - 4xy + 6y^2)^3 \\
&= (3x^2 + 5xy - 5y^2)^3 + (5x^2 - 5xy - 3y^2)^3 \\
&= 8(x - y)(x + y)(x^2 - xy + y^2)(19x^2 - 11xy + 19y^2).
\end{aligned}
\tag{1.5}
$$

It can be shown that there is no third representation in (1.5). Furthermore, (1.4) follows from (1.3) (with the rows permuted) upon making the unimodular linear change of variables: $(x, y) \to (\frac{5x-2y}{\sqrt{21}}, \frac{3x+3y}{\sqrt{21}})$.

Comparable versions of these properties apply to the Narayanan formulas (see (1.14)). More to the point, up to transposition of terms, changes of variable and taking $\lambda \in \mathbb{C}$, we shall show that (1.2) *completely* describes the solution in binary quadratic forms $f_j = f_j(x, y) \in \mathbb{C}[x, y]$ to

$$
p = f_1^3 + f_2^3 = f_3^3 + f_4^3. \tag{1.6}
$$

Our analysis comes from looking at the equation in quadratic forms over $\mathbb{C}$ and studying the properties of the common sum p.

We begin with some notations, following those in [15]. For $m \geq 3$, let $\zeta_m = e^{\frac{2\pi i}{m}}$ and $\omega = \zeta_3$. Two forms in $\mathbb{C}[x, y]$ are *distinct* if they are not proportional. The identity (1.6) is *honest* if the f_j's are pairwise distinct. A *flip* of (1.6) is either of the two equivalent identities

$$
p_1 = f_1^3 - f_3^3 = -f_2^3 + f_4^3, \qquad p_2 = f_1^3 - f_4^3 = -f_2^3 + f_3^3. \tag{1.7}
$$

There seems to be no obvious way of deriving p_1 or p_2 from p in (1.7). If (1.6) holds, we say that the family $\mathcal{F} = \{\{f_1, f_2\}, \{f_3, f_4\}\}$ *represents* p, with the understanding that two families $\mathcal{F}$ and $\mathcal{G}$ are identified if $\{\{f_1^3, f_2^3\}, \{f_3^3, f_4^3\}\} = \{\{g_1^3, g_2^3\}, \{g_3^3, g_4^3\}\}$; we do not care about the order of the summands, or powers of ω multiplying the quadratics. For a sextic form $p \in \mathbb{C}[x, y]$, we define $N(p)$ to be the number of pairwise-nonsimilar families $\mathcal{F}$ representing p.

If $M(x, y) = (\alpha x + \beta y, \gamma x + \delta y)$ is an invertible linear change of variables (or *linear change* for short), and $f \in \mathbb{C}[x, y]$ is a form, define $f \circ M$ by $(f \circ M)(x, y) = f(\alpha x + \beta y, \gamma x + \delta y)$. A *scaling* is a linear change in which $\beta = \gamma = 0$. If $\deg f = d$, and $\delta = \alpha$ in a scaling M, then $f \circ M = \alpha^d f$, If M is a linear change, and $g = f \circ M$, then f and g are *similar*, and if $\mathcal{G} = \mathcal{F} \circ M$, the $\mathcal{F}$ and $\mathcal{G}$ will also be called similar.

More generally, suppose the equation

$$
p = f_1^3 + f_2^3 \tag{1.8}
$$

holds. If M is a linear change, then (1.8) implies that $p\circ M = (f_1\circ M)^3 + (f_2\circ M)^3$. It may happen that $p = p\circ M$, but that $\{(f_1\circ M)^3, (f_2\circ M)^3\} \neq \{f_1^3, f_2^3\}$: this seems to be the inherent mechanism behind multiple representations.

The following underlying identity is central to our analysis. For $\alpha\in\mathbb{C}$,

$$(\alpha x^2 - xy + \alpha y^2)^3 + \alpha(-x^2 + \alpha xy - y^2)^3 = (\alpha^2-1)(\alpha x^3+y^3)(x^3+\alpha y^3). \tag{1.9}$$

(This can easily be verified by setting $v = x^2+y^2$ and $w = xy$ and noting that $v^3 - 3vw^2 = x^6+y^6$.) Observe that the sum is a quadratic in $\{x^3, y^3\}$, and so if $(x,y)\mapsto(\omega x, \omega^2 y)$, then the sum is unchanged, although the summands are changed. Writing $\alpha = \lambda^3$, we can bring in the outside coefficient and obtain

$$\begin{aligned}
&(\lambda^3x^2 - xy + \lambda^3y^2)^3 + (-\lambda x^2 + \lambda^4xy - \lambda y^2)^3\\
&= (\lambda^3\omega^2x^2 - xy + \lambda^3\omega y^2)^3 + (-\lambda\omega^2x^2 + \lambda^4xy - \lambda\omega y^2)^3\\
&= (\lambda^3\omega x^2 - xy + \lambda^3\omega^2 y^2)^3 + (-\lambda\omega x^2 + \lambda^4xy - \lambda\omega^2 y^2)^3\\
&= p_{1,\lambda}(x,y) := (\lambda^6-1)(\lambda^3x^3+y^3)(x^3+\lambda^3y^3).
\end{aligned} \tag{1.10}$$

Write the summands in (1.10) as:

$$\begin{aligned}
F_{1,\lambda}(x,y) &= \lambda^3x^2 - xy + \lambda^3y^2, \quad F_{2,\lambda}(x,y) = -\lambda x^2 + \lambda^4xy - \lambda y^2,\\
F_{3,\lambda}(x,y) &= F_{1,\lambda}(\omega x, \omega^2 y), \quad F_{4,\lambda}(x,y) = F_{2,\lambda}(\omega x, \omega^2 y),\\
F_{5,\lambda}(x,y) &= F_{1,\lambda}(\omega^2 x, \omega y), \quad F_{6,\lambda}(x,y) = F_{2,\lambda}(\omega^2 x, \omega y).
\end{aligned} \tag{1.11}$$

If $\lambda = 0$ or $\lambda^6 = 1$, then the identities of (1.10) are not honest, so we shall assume that $\lambda(\lambda^6-1)\neq 0$. Let $\mathcal{F}_{1,\lambda} = \{\{F_{1,\lambda}, F_{2,\lambda}\}, \{F_{3,\lambda}, F_{4,\lambda}\}\}$, $\mathcal{F}_{2,\lambda} = \{\{F_{3,\lambda}, F_{4,\lambda}\}, \{F_{5,\lambda}, F_{6,\lambda}\}\}$ and $\mathcal{F}_{3,\lambda} = \{\{F_{5,\lambda}, F_{6,\lambda}\}, \{F_{1,\lambda}, F_{2,\lambda}\}\}$. Observe that under the scaling $(x,y)\mapsto(\omega x, \omega^2 y)$, $\mathcal{F}_{1,\lambda}\mapsto\mathcal{F}_{2,\lambda}\mapsto\mathcal{F}_{3,\lambda}\mapsto\mathcal{F}_{1,\lambda}$. Thus the three sets of equations $F_{1,\lambda}^3 + F_{2,\lambda}^3 = F_{3,\lambda}^3 + F_{4,\lambda}^3$, $F_{1,\lambda}^3 + F_{2,\lambda}^3 = F_{5,\lambda}^3 + F_{6,\lambda}^3$, and $F_{3,\lambda}^3 + F_{4,\lambda}^3 = F_{5,\lambda}^3 + F_{6,\lambda}^3$ are similar to each other. The "cleanest" versions of the flips come from $\mathcal{F}_{2,\lambda}$:

$$\begin{aligned}
&F_{4,\lambda}^3(x,y) - F_{5,\lambda}^3(x,y) = -F_{3,\lambda}^3(x,y) + F_{6,\lambda}^3(x,y) = p_{2,\lambda}(x,y) :=\\
&((1+\lambda^6)x^3 + 3\lambda^3x^2y - \lambda^3y^3)(-\lambda^3x^3 + 3\lambda^3xy^2 + (1+\lambda^6)y^3);
\end{aligned} \tag{1.12}$$

$$\begin{aligned}
&F_{4,\lambda}^3(x,y) - F_{6,\lambda}^3(x,y) = F_{5,\lambda}^3(x,y) - F_{3,\lambda}^3(x,y) = p_{3,\lambda}(x,y) :=\\
&3\sqrt{-3}\, xy(x-y)(x+y)(\lambda^3x+y)(x+\lambda^3y).
\end{aligned} \tag{1.13}$$

We now present some symmetries of (1.10). Since $F_{j,-\lambda}(x,y) = -F_{j,\lambda}(x,-y)$, $\mathcal{F}_{j,-\lambda}$ is similar to $\mathcal{F}_{j,\lambda}$. Further, $F_{1,\lambda^{-1}} = -\lambda^{-4}F_{2,\lambda}$ and $F_{2,\lambda^{-1}} = -\lambda^{-4}F_{1,\lambda}$, etc., so $\mathcal{F}_{j,\lambda^{-1}}$ is similar to $\mathcal{F}_{j,\lambda}$. Under the unimodular linear change

$$(x,y)\mapsto \tfrac{1}{\sqrt{1-\lambda^6}}(\lambda^3x+y, -(x+\lambda^3y)),$$

the system of identities

$$F_{1,\lambda}^3(x,y) + F_{2,\lambda}^3(x,y) = F_{3,\lambda}^3(x,y) + F_{4,\lambda}^3(x,y) = F_{5,\lambda}^3(x,y) + F_{6,\lambda}^3(x,y)$$

becomes

$$\begin{aligned} F^3_{7,\lambda}(x,y)+F^3_{8,\lambda}(x,y) &= -F^3_{3,\lambda}(x,y)+F^3_{6,\lambda}(x,y) = -F^3_{5,\lambda}(x,y)+F^3_{4,\lambda}(x,y);\\ F_7(x,y) &= \tfrac{1}{1-\lambda^6}\left((2\lambda^3+\lambda^9)x^2+(1+5\lambda^6)xy+(2\lambda^3+\lambda^9)y^2\right),\\ F_8(x,y) &= -\tfrac{\lambda}{1-\lambda^6}\left((1+2\lambda^6)x^2+(5\lambda^3+\lambda^9)xy+(1+2\lambda^6)y^2\right). \end{aligned}$$

Of course, $p_1(x,y)\mapsto p_2(x,y)$ under this linear change. This means that each $\mathcal{F}_{j,\lambda}$ is similar to one of its flips.

If we make the linear change $(x,y)\mapsto(x+\omega^2y,x+\omega y)$ into (1.10), we obtain an enhanced version of (1.2), with a third sum:

$$\begin{aligned} &N^3_{4,\lambda}(x,y)+N^3_{3,\lambda}(x,y) = -N^3_{2,\lambda}(x,y)+N^3_{1,\lambda}(x,y)\\ &= (-px^2+(m+2p)xy-py^2)^3+(\ell x^2+(n-2\ell)xy+\ell y^2)^3. \end{aligned} \tag{1.14}$$

Upon continuing with the linear change which takes (1.10) into (1.12), we get a flipped version of (1.2) and another third equal sum, but with denominators. A slightly different linear change gives a simple version in $\mathbb{Q}(\lambda)[x,y]$: under $(x,y)\mapsto(x-\sqrt{-3}\;y,x+\sqrt{-3}\;y)$, and multiplication by -1, (1.10) becomes

$$\begin{aligned} &((1-2\lambda^3)x^2+3(1+2\lambda^3)y^2)^3+(\lambda(2-\lambda^3)x^2-3\lambda(2+\lambda^3)y^2)^3\\ &=((1+\lambda^3)x^2+6\lambda^3xy+3(1-\lambda^3)y^2)^3+(-\lambda(1+\lambda^3)x^2-6\lambda xy+3\lambda(1-\lambda^3)y^2)^3\\ &=((1+\lambda^3)x^2-6\lambda^3xy+3(1-\lambda^3)y^2)^3+(-\lambda(1+\lambda^3)x^2+6\lambda xy+3\lambda(1-\lambda^3)y^2)^3. \end{aligned}$$

It is also worth noting that under the linear change $(x,y)\mapsto(x+\tau y,-i(\tau x-y))$, $\tau=\sqrt{1-\lambda^6}-i\lambda^3$, (which is invertible provided $\lambda^6\neq1$), (1.13) becomes an equation of the shape $(ax^2+bxy+ay^2)^3+(ax^2-bxy+ay^2)^3=(rx^2+sy^2)^3+(sx^2+ry^2)^3$, and $p_{3,\lambda}$ becomes a multiple of $x^6+(4\lambda^6-1)x^4y^2+(4\lambda^6-1)x^2y^4+y^6$. This phenomenon is explored in Theorem 3.1.

This paper has two parts. The main result of the first part is the following theorem.

Theorem 1.1. *Every honest identity* (1.6) *for binary sextics is similar to some* $\mathcal{F}_{2,\lambda}$ *with* $\lambda(\lambda^6-1)\neq0$*, up to a possible flip.*

There is a crucial intermediate step in the proof of Theorem 1.1. Any four binary quadratic forms are linearly dependent, and a given dependence is not affected by a linear change. We shall say that an honest (1.6) is an identity of *Type*(T) for $T\in\mathbb{C}$ if, perhaps after a flip, the following two equations hold:

$$f_1^3+f_2^3=f_3^3+f_4^3,\qquad f_1+f_2=T(f_3+f_4). \tag{1.15}$$

We show (see Lemma 4.2) that $T(T^3-1)\neq0$ in an honest family of Type(T). Of course, the same equation is both Type(T) and Type(T^{-1}), and factors of ω^k do not matter.

The following identities show that (1.2) and (1.9) are both of Type(λ^2):

$$\begin{aligned} N^3_{2,\lambda}+N^3_{4,\lambda} &= N^3_{1,\lambda}-N^3_{3,\lambda},\quad N_{2,\lambda}+N_{4,\lambda}=\lambda^2(N_{1,\lambda}-N_{3,\lambda});\\ F^3_{5,\lambda}-F^3_{3,\lambda} &= F^3_{4,\lambda}-F^3_{6,\lambda},\quad F_{5,\lambda}-F_{3,\lambda}=\lambda^2(F_{4,\lambda}-F_{6,\lambda}). \end{aligned}$$

We prove Theorem 1.1 in two stages. After a few technical lemmas, we show that after a linear change, for any honest solution (1.6), f_1 and f_2 are both even and that f_3 and f_4 are not (see Corollary 2.5). We then determine all honest (1.6) in which f_3, f_4 are not both even, but $f_3^3 + f_4^3$ is (see Theorems 3.1, 3.2) and show that they must be of Type(T) for some T. (Geometrically, this says that any quadratic curve which lies on the surface $z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0$ must in fact lie on the intersection of the surface with a hyperplane $z_i + z_j + T(z_k + z_\ell) = 0$ for some permutation of the indices.) We finally show that any two honest solutions of (1.6) of Type(T) are similar (Theorem 4.3), and are similar to (1.10) (or (1.2)) with $T = \lambda^2$. Sándor [18] proved in 1996, using different methods, that any solution to (1.6) must be Type(T) for some T, and that two such solutions in $\mathbb{Q}[x, y]$ with the same T must be similar.

We also explore solutions to (1.6) with $f_j \in \mathbb{Q}[x, y]$. If such an equation has type $T = \lambda^2$, then it is clear from linear algebra that $T \in \mathbb{Q}$. If $\lambda \in \mathbb{Q}$, then (1.14) lies in $\mathbb{Q}[x, y]$. The preprint version of this paper contained a conjecture that this condition is necessary, as well as sufficient. The referee pointed out that this was not correct, with suggestions to the correct result and its proof. We show at the end of the paper (Theorem 7.4) that the solutions in $\mathbb{Q}[x, y]$ occur precisely for T which are $a^2 + 3b^2$ for $a, b \in \mathbb{Q}$. Sándor had also proved this, in a different way, in [18].

In the second part of the paper, we give a complete description of $N(p)$, the number of different ways that a binary sextic form is a sum of two cubes. A key result (see Theorem 5.1) is that a form p (of degree $3k$) is a sum of two cubes if and only if $p = h_1h_2h_3$ where the h_j's are distinct, but linearly dependent. There are two important families of sextics: for $t \in \mathbb{C}$, let

$$A_t(x, y) := x^6 + tx^4y^2 + tx^2y^4 + y^6, \qquad B_t(x, y) := x^6 + tx^3y^3 + y^6. \tag{1.16}$$

Observe that $p_{1,\lambda} = \lambda^3(\lambda^6 - 1)B_{\lambda^3+\lambda^{-3}}$, and as we have seen, $p_{3,\lambda}$ is similar to $A_{4\lambda^6-1}$. Every A_t and B_t is thus similar to $p_{1,\lambda}$ or $p_{3,\lambda}$ for λ with $\lambda(1 - \lambda^6) \neq 0$ except for $A_{-1}, A_3, B_{\pm 2}$.

We give a census of $N(p)$ for binary sextics: (i) a binary sextic p is a sum of two cubes (that is, $N(p) \geq 1$) if and only if $p = \ell^3 q$, where ℓ is linear and q is a square-free cubic or p is similar to $q(x^2, y^2)$, where q is a square-free cubic (see Theorem 5.3); (ii) a binary sextic p has $N(p) = 2$ if and only if p is similar to A_t for $t \in \mathbb{C}$, except that $N(A_3) = 0$, $N(A_{-1}) = 1$, $N(A_0) = N(A_{15}) = 4$ and $N(A_{-5}) = 6$ (see Theorem 5.4); (iii) a binary sextic p has $N(p) = 3$ if and only if p is similar to B_t for $t \in \mathbb{C}$, except that $N(B_{\pm 2}) = 0$, $N(B_0) = 4$ and $N(B_{\pm 5\sqrt{-2}}) = 6$, (see Theorem 5.5); (iv) up to similarity, there are two sextics with $N(p) > 3$:

$$Q_1(x, y) = x^6 + y^6 \quad \text{or} \quad Q_2(x, y) = xy(x^4 - y^4). \tag{1.17}$$

To be specific, Q_1 is similar to A_0, A_{15}, B_0 and $N(Q_1) = 4$ and Q_2 is similar to A_{-5} and $B_{\pm 5\sqrt{-2}}$ and $N(Q_2) = 6$ (see Theorem 5.6). Section six gives some extra attention to the representations of Q_1, Q_2 and their similarities.

In the final section, we give some different directions that this study might go. We show that the classical Euler-Binet parameterization of equal sums of two cubes over

$\mathbb{Q}$ is also valid over $\mathbb{C}(x_1, \dots, x_n)$ (see Theorem 7.1 and Corollary 7.2). We apply the usual "point addition" of points on the curve $x_1^3 + x_2^3 = x_3^3 + x_4^3 = A$ to show that $(F_{1,\lambda}, F_{2,\lambda})$"+"$(F_{3,\lambda}, F_{4,\lambda}) = (F_{5,\lambda}, F_{6,\lambda})$ (see Theorem 7.3); the denominators disappear. We show, separately, that a flip of the Euler-Binet parameterization can be added to find a third representation as a sum of cubes of polynomials (see (7.14).) Finally, we present a few previous results from the huge literature, including those of Sándor [18], which when combined with Euler-Binet, allows us a quick proof of Theorem 7.4. Besides [18], we have not found a systematic analysis of (1.6) over $\mathbb{C}[x, y]$, and we have not found (1.9) or, indeed, any three-fold identities.

This project began 20 years ago when Bruce Berndt gave a seminar at Illinois about (1.1) and (1.2). The author foolishly believed that an algebraic approach would easily lead to all solutions, and posted a proof-free online set of notes [13] in 2000. Eventually, it has produced this article and an earlier companion paper studying higher powers, [15]. He wishes to thank his present and former colleagues Michael Bennett, Bruce Berndt, Nigel Boston, Dan Grayson and Jeremy Rouse for helpful conversations, and Andrew Bremner, Noam Elkies and Michael Hirschhorn for encouraging and useful emails over the years. He also wants to thank the referee for a careful and sympathetic reading of the manuscript, as well as suggesting the key to the proof of Theorem 7.4.

2. Preliminary lemmas

We begin with several old simple lemmas, giving proofs for completeness. The first is a special case of, for example, [15, Thm.1.1].

Lemma 2.1. *If* $\{\alpha_k x + \beta_k y\}, 1 \le k \le r \le 4$, *are pairwise distinct linear forms, then* $\{(\alpha_k x + \beta_k y)^3\}$ *is linearly independent. In particular, if* (1.6) *holds and* $\{h_j\}$ *is honest, then it cannot be the case that each* h_j *is even.*

Proof. If $r < 4$, add more distinct linear forms to assume that $r = 4$. The matrix of $\{(\alpha_j x+\beta_j y)^3\}$ with respect to the basis $\{\binom{3}{i} x^{3-i} y^i\}$ is $[\alpha_j^{3-i}\beta_j^i]$, which is Vandermonde, with determinant $\prod_{1\le j<k\le 4}(\alpha_j\beta_k - \alpha_k\beta_j)$. This determinant is non-zero because each pair of linear forms is distinct.

Suppose p is a cubic form and

$$p(x, y) = (\alpha_1 x + \beta_1 y)^3 + (\alpha_2 x + \beta_2 y)^3 = (\alpha_3 x + \beta_3 y)^3 + (\alpha_4 x + \beta_4 y)^3. \tag{2.1}$$

Then $0 = p - p$ gives a formal linear dependence of four cubics, which must result from pairwise cancellation; that is, the original representations were the same.

Finally, by comparing coefficients, the equation

$$(\alpha_1 x^2 + \beta_1 y)^3 + (\alpha_2 x^2 + \beta_2 y^2)^3 = (\alpha_3 x^2 + \beta_3 y^2)^3 + (\alpha_4 x^2 + \beta_4 y^2)^3$$

implies (2.1), and so cannot happen in an honest family. □

Lemma 2.2. *Suppose* $g_1, g_2 \in \mathbb{C}[x_1, \dots, x_n]$ *are distinct forms. Then for* $d \ge 2$, *the set* $\{g_1^{d-k} g_2^k : 0 \le k \le d\}$ *is linearly independent.*

Proof. Suppose $\sum_{k=0}^{d} \lambda_k g_1^{d-k} g_2^k = 0$ for a non-zero choice of $\{\lambda_k\}$. Then

$$\sum_{k=0}^{d} \lambda_k x^{d-k} y^k = \prod_{j=1}^{d} (\alpha_j x + \beta_j y) \implies \prod_{j=1}^{d} (\alpha_j g_1 + \beta_j g_2) = 0;$$

thus $\alpha_j g_1 + \beta_j g_2 = 0$ for some j, violating the distinctness hypothesis. □

We need an old fact about simultaneous diagonalization; there doesn't seem to be a standard easy-to-find modern proof, a different proof is shown in [15, Thm.3.2].

Theorem 2.3. *If $f_1(x, y)$ and $f_2(x, y)$ are relatively prime quadratic forms, then there is a linear change M so that $f_1 \circ M$ and $f_2 \circ M$ are both even.*

Proof. We may assume $rank(f_1) \ge rank(f_2) \ge 1$, and after a preliminary linear change, take $f_1(x, y) = x^2$ or $x^2 + y^2$. In the first case, $rank(f_2) = 1$, so $f_2 = \ell^2$ for a linear ℓ which can become y after a linear change, so $(f_1, f_2) \to (x^2, y^2)$. Otherwise, we have $f_1(x, y) = x^2 + y^2$ and $f_2(x, y) = ax^2 + 2bxy + cy^2$. Since f_1 and f_2 are relatively prime, $x \pm iy$ is not a factor of f_2 and so $a \pm 2bi - c \neq 0$.

The quadratic $\lambda f_1 + f_2$ has discriminant

$$\Delta(\lambda) = 4(\lambda + a)(\lambda + c) - (2b)^2 = 4(\lambda^2 + (a + c)\lambda + (ac - b^2));$$

$$Disc(\Delta(\lambda)) = 16((a + c)^2 - 4(ac - b^2)) = 16(a + 2bi - c)(a - 2bi - c) \neq 0.$$

Thus there exist $\lambda_1 \neq \lambda_2$ so that each quadratic $\lambda_j f_1 + f_2$ is perfect square; that is, $\lambda_j f_1 + f_2 = \ell_j^2$. This implies that both f_1 and f_2 are linear combinations of ℓ_1^2, ℓ_2^2. A linear change taking $(\ell_1, \ell_2) \mapsto (x, y)$ completes the diagonalization. □

In order to apply Theorem 2.3, we need a small technical lemma.

Lemma 2.4. *Suppose $p = f_1^3 + f_2^3 = f_3^3 + f_4^3$ for quadratic f_1, f_2 and f_1 and f_2 have a non-trivial common factor. Then $\{f_1^3, f_2^3\} = \{f_3^3, f_4^3\}$. Thus in any honest instance of* (1.6)*, the f_j's are pairwise relatively prime.*

Proof. Suppose ℓ is a linear form and $f_1 = \ell\ell_1$ and $f_2 = \ell\ell_2$. Then

$$\ell^3 \mid f_3^3 + f_4^3 = (f_3 + f_4)(f_3 + \omega f_4)(f_3 + \omega^2 f_4).$$

Since the three factors on the right are quadratic, ℓ must divide at least two of them; it follows that ℓ divides both f_3 and f_4. By writing $f_3 = \ell\ell_3$ and $f_4 = \ell\ell_4$, we see that $\ell_1^3 + \ell_2^3 = \ell_3^3 + \ell_4^3$, and since the original equation was honest, the ℓ_j's are pairwise distinct. This is impossible by Lemma 2.1. □

Putting the results of this section together, we have the following corollary.

Corollary 2.5. *If an honest* (1.6) *holds, then after a linear change, f_1 and f_2 are even, (and hence so is p), but f_3 and f_4 are not both even; thus*

$$(ax^2 + bxy + cy^2)^3 + (dx^2 + exy + fy^2)^3 \tag{2.2}$$

is even, where $(b, e) \neq (0, 0)$.

3. Even sums of the cubes of non-even quadratic forms

Our goal in this section is to show that every quadratic solution to (1.6) is a family of Type(T) for some T.

How can it happen that $f_3^3 + f_4^3$ is even when at least one of $\{f_3, f_4\}$ is not even? An obvious case is

$$f_3(x,y) = ax^2 + bxy + cy^2, \qquad f_4(x,y) = ax^2 - bxy + cy^2, \tag{3.1}$$

which, as in [15], we call the *tame* case; otherwise we are in the *wild* case. If $a = 0$, then it follows from (3.1) that y divides f_3 and f_4, and by Lemma 2.4, this cannot happen, so $a \neq 0$. Similarly, $c \neq 0$. Thus, we may scale x and y and assume that f_3, f_4 are $x^2 \pm \gamma xy + y^2$ for some $\gamma \neq 0$.

Theorem 3.1. *The tame case occurs in a family of Type*$((1 + \frac{3}{4}\gamma^2)^{1/3})$.

Proof. Observe that

$$\begin{gathered}(x^2 + \gamma xy + y^2)^3 + (x^2 - \gamma xy + y^2)^3 = \\ 2(x^6 + 3(1+\gamma^2)x^4y^2 + 3(1+\gamma^2)x^2y^4 + y^6) = 2A_{3(1+\gamma^2)}(x,y).\end{gathered} \tag{3.2}$$

Let $\{f_3(x,y), f_4(x,y)\} = \{x^2 \pm \gamma xy + y^2\}$. Honesty requires $\gamma \neq 0$. By hypothesis, $2A_{3(1+\gamma^2)}$ is a sum of cubes of two even quadratics in a unique way by Lemma 2.1.

Note that (3.2) implies that

$$\begin{gathered}2(x^6 + 3(1+\gamma^2)x^4y^2 + 3(1+\gamma^2)x^2y^4 + y^6) = (r_\gamma x^2 + s_\gamma y^2)^3 + (s_\gamma x^2 + r_\gamma y^2)^3 \\ \iff r_\gamma^3 + s_\gamma^3 = 2, \quad 3r_\gamma^2 s_\gamma + 3r_\gamma s_\gamma^2 = 3r_\gamma s_\gamma(r_\gamma + s_\gamma) = 6(1+\gamma^2) \\ \implies (r_\gamma + s_\gamma)^3 = 8 + 6\gamma^2.\end{gathered} \tag{3.3}$$

Observe that if $\gamma^2 = -\frac{4}{3}$, then $0 = (r_\gamma + s_\gamma)^3$, so $s_\gamma = -r_\gamma$ and $r_\gamma^3 + s_\gamma^3 = 0$, so we take $\gamma^2 \neq -\frac{4}{3}$. Up to $(r_\gamma, s_\gamma) \mapsto \omega^k(r_\gamma, s_\gamma)$ and a choice of cube root,

$$r_\gamma + s_\gamma = (8 + 6\gamma^2)^{1/3} \neq 0 \implies r_\gamma s_\gamma = \frac{2(1+\gamma^2)}{(8+6\gamma^2)^{1/3}},$$

and so r_γ and s_γ are the roots of the quadratic equation

$$X^2 - (8+6\gamma^2)^{1/3}X + \frac{2(1+\gamma^2)}{(8+6\gamma^2)^{1/3}} = 0.$$

Let $\{f_1(x,y), f_2(x,y)\} = \{r_\gamma x^2 + s_\gamma y^2, s_\gamma x^2 + r_\gamma y^2\}$. Since $(r_\gamma - s_\gamma)^2 = (r_\gamma + s_\gamma)^2 - 4r_\gamma s_\gamma = -\frac{2\gamma^2}{(8+6\gamma^2)^{1/3}} \neq 0$, these roots are distinct, and since $(r_\gamma + s_\gamma)(f_3 + f_4) = 2(f_1 + f_2)$, the equation $f_1^3 + f_2^3 = f_3^3 + f_4^3$ is a Type$(\frac{r_\gamma + s_\gamma}{2})$ family. □

Theorem 3.2. *If*

$$p(x,y) = f_1^3(x,y) + f_2^3(x,y) := (ax^2 + bxy + cy^2)^3 + (dx^2 + exy + fy^2)^3 \tag{3.4}$$

is even and a sum of two even cubes $f_3^3(x,y)+f_4^3(x,y)$, $(b,e)\neq(0,0)$, *and* $(d,e,f)\neq\omega^k(a,-b,c)$, *then a flip of* $f_1^3+f_2^3=f_3^3+f_4^3$ *is a Type*(T) *family for some* T *and* p *has a third representation as a sum of two cubes.*

Proof. By considering the coefficients of x^5y, x^3y^3, xy^5 in (3.4), we need to solve

$$3a^2b+3d^2e=6abc+b^3+6def+e^3=3bc^2+3ef^2=0. \tag{3.5}$$

If $a=0$ in (3.5), then $d^2e=0$. If $d=0$, then $a=d=0$ implies a common factor in the quadratics, violating Lemma 2.4. Hence $a=e=0$, so $b^3=0$ and $b=e=0$. These contradictions imply that $a\neq 0$; similar arguments show that $cef\neq 0$. And now, if $b=0$, then $d^2e=0$ and $e\neq 0$ imply $d=0$, so $b\neq 0$ after all. Similarly $e\neq 0$. Thus all variables in (3.5) are non-zero.

By a scaling of (x,y), we may assume $a=c=1$, so

$$p(x,y)=(x^2+bxy+y^2)^3+(dx^2+exy+fy^2)^3 \tag{3.6}$$

is even, and (3.5) becomes

$$3b+3d^2e=6b+b^3+6def+e^3=3b+3f^2e=0. \tag{3.7}$$

It follows that $b=-d^2e$ and $f^2=d^2$; the remaining equation becomes

$$0=-6d^2e-d^6e^3+6def+e^3=e^3(1-d^6)+6de(f-d). \tag{3.8}$$

If $f=d$ in (3.8), then $d^6=1$, so up to a power of ω, $d\in\{1,-1\}$. If $d=1$, then $e=-b, f=1$ implies that (3.6) is tame; if $d=-1$, then $e=-b, f=-1$ implies that $p=0$. In the remaining case, $f=-d$ and $e^2(1-d^6)=12d^2$, so $e=\pm\frac{2\sqrt{3}d}{\sqrt{1-d^6}}$, $d^6\neq 1$. By taking $y\mapsto -y$ if necessary, we may choose one square root and rewrite (3.6) as

$$p(x,y)=\left(x^2-\frac{2\sqrt{3}d^3}{\sqrt{1-d^6}}\,xy+y^2\right)^3+\left(dx^2+\frac{2\sqrt{3}d}{\sqrt{1-d^6}}\,xy-dy^2\right)^3. \tag{3.9}$$

Write (3.9) as $p=f_1^3+f_2^3$. Pull d^3 out of the second factor and let $r=d^3$. A computation shows that

$$p(x,y)=(1+r)x^6+\frac{3(1+10r+r^2)}{1-r}\,x^4y^2+\frac{3(1-10r+r^2)}{1+r}\,x^2y^4+(1-r)y^6.$$

We use the Sylvester algorithm (see [14, Thm.2.1]) to write p as a sum of two cubes of even quadratics. In this way, and omitting details, we find that

$$\begin{aligned}p(x,y)=r\left(-\frac{2+3r+r^2}{1-r^2}\cdot x^2+\frac{2-3r+r^2}{1-r^2}\cdot y^2\right)^3&+\\ \left(\frac{1+3r+2r^2}{1-r^2}\cdot x^2+\frac{1-3r+2r^2}{1-r^2}\cdot y^2\right)^3&.\end{aligned} \tag{3.10}$$

Write (3.10) as $f_3^3 + f_4^3$, and restore $r = d^3$, so we now have

$$\begin{aligned} f_1(x,y) &= x^2 - \frac{2\sqrt{3}d^3}{\sqrt{1-d^6}}xy + y^2, \quad f_2(x,y) = dx^2 + \frac{2\sqrt{3}d}{\sqrt{1-d^6}}xy - dy^2, \\ f_3(x,y) &= -\frac{d(2+3d^3+d^6)}{1-d^6}\cdot x^2 + \frac{d(2-3d^3+d^6)}{1-d^6}\cdot y^2, \\ f_4(x,y) &= \frac{1+3d^3+2d^6}{1-d^6}\cdot x^2 + \frac{1-3d^3+2d^6}{1-d^6}\cdot y^2. \end{aligned} \tag{3.11}$$

Putting this together, (3.9), (3.10) and (3.11) imply that

$$f_1^3(x,y) - f_4^3(x,y) = f_3^3(x,y) - f_2^3(x,y),$$
$$f_1(x,y) + d^2 f_2(x,y) = d^2 f_3(x,y) + f_4(x,y) \quad (= (1+d^3)x^2 + (1-d^3)y^2).$$

Thus, the wild case flips into a Type(d^2) family. Since p is even, and f_3, f_4 are not, we also have $p = f_5^3 + f_6^3$ where $f_5(x,y) = f_3(x,-y)$ and $f_6(x,y) = f_4(x,-y)$. □

4. Equations of Type(T)

In this section we completely describe the solutions to (1.6) of Type(T). We begin with a probably familiar result from Diophantine analysis.

Proposition 4.1. *Suppose $\Phi(u,v) = au^2 + 2buv + cv^2$ is a rank two quadratic form in $\mathbb{C}[x,y]$. Then any two honest solutions $(p_i, q_i, r_i), i = 1, 2$, in binary quadratic forms to the following equation are similar.*

$$\Phi(p,q) = r^2. \tag{4.1}$$

Proof. Write (4.1) as $(a_{11}p + a_{12}q)(a_{21}p + a_{22}q) = r^2$, where the factors on the left are distinct. Since $\gcd(p,q) = 1$, $\gcd(a_{11}p + a_{12}q, a_{21}p + a_{22}q) = 1$ as well. It follows by unique factorization that $(a_{11}p + a_{12}q, a_{21}p + a_{22}q, r) = (g^2, h^2, gh)$, for suitable distinct linear forms g, h. Let $[b_{ij}] = [a_{ij}]^{-1}$. Then

$$(p,q,r) = (b_{11}g^2 + b_{12}h^2, b_{21}g^2 + b_{22}h^2, gh).$$

In particular, (p_j, q_j, r_j) comes from (g_j, h_j), and the linear change M taking the honest pairs of linear forms (g_1, h_1) into (g_2, h_2) will take (p_1, q_1, r_1) into (p_2, q_2, r_2). □

Lemma 4.2. *If* (1.6) *is honest and a Type(T) family, then* $T(T^3 - 1) \neq 0$.

Proof. If $T = 0$, then $f_2 = -f_1$, violating honesty. Suppose $T^3 = 1$, so $T = \omega^k$. Then by $(f_3, f_4) \mapsto \omega^k(f_3, f_4)$ we may assume that $f_1 + f_2 = f_3 + f_4$. In this case, we have

$$(f_1 + f_2)^2 - \frac{f_1^3 + f_2^3}{f_1 + f_2} = (f_3 + f_4)^2 - \frac{f_3^3 + f_4^3}{f_3 + f_4} \implies f_1 f_2 = f_3 f_4. \tag{4.2}$$

This implies that $\{f_1, f_2\} = \{f_3, f_4\}$, again violating honesty. □

Theorem 4.3. *Suppose* $\{f_1, f_2, f_3, f_4\}$ *is an honest Type*(T) *family; specifically*

$$
\begin{aligned}
f_1^3 + f_2^3 &= f_3^3 + f_4^3, \\
f_1 + f_2 = T(f_3 + f_4), &\qquad T(T^3 - 1) \neq 0,
\end{aligned} \tag{4.3}
$$

and let $T = \lambda^2$. *Then there is a linear change* M *so that* $\{f_1 \circ M, f_2 \circ M\} = \{F_{3,\lambda}, -F_{5,\lambda}\}$ *and* $\{f_3 \circ M, f_4 \circ M\} = \{-F_{4,\lambda}, F_{6,\lambda}\}$.

Proof. As in (4.2), after dividing the equations in (4.3) we obtain

$$f_1^2 - f_1 f_2 + f_2^2 = T^{-1}(f_3^2 - f_3 f_4 + f_4^2). \tag{4.4}$$

It follows that

$$
\begin{aligned}
3f_1 f_2 = (f_1 + f_2)^2 - (f_1^2 - f_1 f_2 + f_2^2) = \\
(T^2 - T^{-1})f_3^2 + (2T^2 + T^{-1})f_3 f_4 + (T^2 - T^{-1})f_4^2.
\end{aligned} \tag{4.5}
$$

But f_1 and f_2 are quadratic forms, and also the roots of the quadratic

$$
\begin{aligned}
&(X - f_1)(X - f_2) = \\
&X^2 - T(f_3 + f_4)X + \tfrac{1}{3}((T^2 - T^{-1})(f_3^2 + f_4^2) + (2T^2 + T^{-1})f_3 f_4) \\
&\implies \{f_1, f_2\} = \{\tfrac{T}{2}(f_3 + f_4) \pm \tfrac{1}{2}\sqrt{\Delta}\}; \\
&\Delta = \tfrac{1}{3T}\left((4 - T^3)f_3^2 - (4 + 2T^3)f_3 f_4 + (4 - T^3)f_4^2\right) = (f_2 - f_1)^2.
\end{aligned} \tag{4.6}
$$

Consider now the quadratic form Φ, which has rank 2 if $T^3 \neq 1$.

$$\Phi(u, v) = \frac{1}{3T}\left((4 - T^3)u^2 - (4 + 2T^3)uv + (4 - T^3)v^2\right).$$

We have seen that $\Phi(f_3, f_4) = (f_2 - f_1)^2$. It may be checked that

$$\Phi(-F_{4,\lambda}, F_{6,\lambda}) = (\lambda^3 x^2 + 2xy + \lambda^3 y^2)^2. \tag{4.7}$$

Thus by Proposition 4.1, there is a linear change M so that $f_3 = -F_{4,\lambda} \circ M$ and $f_4 = F_{6,\lambda} \circ M$. It is routine to check that the quadratic equation (4.6) then solves to give $\{f_1 \circ M, f_2 \circ M\} = \{F_{3,\lambda}, -F_{5,\lambda}\}$. □

Proof of Theorem 1.1. Combine Theorems 3.1, 3.2 and 4.3. □

The historical motivation for the study of (1.6) was to find parameterizations of equal sums of pairs of rational cubes, so there is a special interest in solutions to (1.6) in which $f_j \in \mathbb{Q}[x, y]$. Since every solution to (1.6) is a Type(T) family for $T = \lambda^2$, we know that if $T = a^2, a \in \mathbb{Q}$, then there is a Type$(T)$ family in $\mathbb{Q}[x, y]$. In fact, there is such a family if and only if $T = a^2 + 3b^2$ for $a, b \in \mathbb{Q}$. We are indebted to the referee for suggesting this result and giving strong directions towards its proof, using [18]. For expository reasons, we defer Theorem 7.4 to the end of the paper.

Finally, a 1595 identity of Vieta (see [16]) becomes a version of (1.6) upon clearing denominators:

$$(x(x^3 - y^3))^3 + (y(x^3 - y^3))^3 = (x(x^3 + 2y^3))^3 + (-y(2x^3 + y^3))^3;$$

the four quartics above are linearly independent. It seems unlikely that the methods of this paper are helpful when f_j in (1.6) have degree greater than two.

5. How many ways is a sextic a sum of two cubes?

We turn to a more general question. Lundqvist, Oneto, Shapiro and the author proved in [9] that every binary sextic in $\mathbb{C}[x,y]$ can be written in infinitely many different ways as a sum of three cubes of quadratic forms. It is natural to wonder which binary sextics can be written as a sum of two cubes, and in how many ways.

We need some more general notation: for distinct forms $F, G \in \mathbb{C}[x_1 \dots, x_n]$, write $X = \langle F, G\rangle$ for the linear subspace $\{c_1F + c_2G\}$, and write $X^3 = \langle F^3, F^2G, FG^2, G^3\rangle$; X^3 is the set of all $h(F,G)$ for binary cubic forms h.

Theorem 5.1. *A form $p \in \mathbb{C}[x_1 \dots, x_n]$ of degree $3r$ can be written as $p = f_1^3 + f_2^3$ for distinct forms f_i of degree r if and only if it has a factorization $p = g_1g_2g_3$ in which the g_k's are distinct but linearly dependent and $\langle f_1, f_2\rangle = \langle g_1, g_2, g_3\rangle$. If p belongs to m different subspaces $\langle F_j, G_j\rangle^3$ as above, then $N(p) \le m$. If p is not divisible by the square of a form of degree r, then $N(p) = m$.*

Proof. In one direction,

$$p = f_1^3 + f_2^3 \implies p = (f_1 + f_2)(f_1 + \omega f_2)(f_1 + \omega^2 f_2) := g_1g_2g_3. \tag{5.1}$$

If any two of the g_i's are proportional in (5.1), then so are f_1 and f_2, and p is a cube contrary to hypothesis. For dependence, $g_j \in \langle f_1, f_2\rangle$, also, $g_1 + \omega g_2 + \omega^2 g_3 = 0$.

Conversely if $P = g_1g_2g_3$ and g_1 and g_2 are distinct with $g_3 \in X = \langle g_1, g_2\rangle$, there exist $\alpha, \beta \ne 0$ so that $g_3 = \alpha g_1 + \beta g_2$. The sum of two cubes follows from an old formula (recall that $\omega - \omega^2 = \sqrt{-3}$):

$$\begin{gathered} p = g_1g_2g_3 = g_1g_2(\alpha g_1 + \beta g_2) = \\ \frac{1}{3\sqrt{-3}\,\alpha\beta} \cdot \left((\omega\alpha g_1 - \beta g_2)^3 + (-\alpha g_1 + \omega\beta g_2)^3\right). \end{gathered} \tag{5.2}$$

Suppose p had two different expressions as a sum of two cubes of forms in $\langle f_1, f_2\rangle$:

$$p = (c_{1,1}f_1 + c_{2,1}f_2)^3 + (c_{3,1}f_1 + c_{4,1}f_2)^3 = (c_{1,2}f_1 + c_{2,2}f_2)^3 + (c_{3,2}f_1 + c_{4,2}f_2)^3.$$

Then by the linear independence of $\{f_1^{3-k}f_2^k\}$ from Lemma 2.2, it follows that

$$(c_{1,1}x + c_{2,1}y)^3 + (c_{3,1}x + c_{4,1}y)^3 = (c_{1,2}x + c_{2,2}y)^3 + (c_{3,2}x + c_{4,2}y)^3,$$

which contradicts Lemma 2.1.

Thus, every representation of $p = f_1^3 + f_2^3$ identifies the subspace $\langle f_1, f_2\rangle^3$. Conversely, if $p \in \langle f_1, f_2\rangle^3$, then there is a cubic form h so that $p = h(f_1, f_2)$ and

$$h(x,y) = \sum_{j=1}^{2}(\alpha_j x + \beta_j y)^3 \implies p(x,y) = \sum_{j=1}^{2}(\alpha_j f_1 + \beta_j f_2)^3.$$

If $p \in \langle f_1, f_2\rangle^3$, then p is a sum of two cubes, unless h is a cube (and hence so is p), or $h(x,y) = (\alpha_1 x + \beta_1 y)^2(\alpha_2 x + \beta y)$, so p is divisible by $(\alpha_1 f_1 + \beta_1 f_2)^2$. □

Our study of sextics relies critically on the behavior of cubics as a sum of cubes. An important corollary was known in the 19th century (see also e.g. [14, Thm.5.2]). A binary cubic q is *square-free* if it is a product of three pairwise distinct linear factors.

Proposition 5.2. *If p is a binary cubic which is not the cube of a linear form, then $p = \ell_1^3 + \ell_2^3$ for distinct linear forms ℓ_j if and only if p it is square-free, and this representation is unique,*

Proof. In the general case, $f = \ell_1\ell_2\ell_3$ is a product of three distinct linear forms; any three such forms are linearly dependent. The other cases are $f = \ell^3$ and $f = \ell_1^2\ell_2$, and the necessary factorization is impossible. □

For Theorems 5.4, 5.5, 5.6, recall (1.17).

Theorem 5.3. *A binary sextic $p(x, y)$ is an honest sum of two cubes ($N(p) \geq 1$) if and only if one of the two conditions hold: (i) $p = \ell^3 q$, where ℓ is linear form and q is a square-free cubic; or (ii) p is similar to $q(x^2, y^2)$, where q is a square-free cubic, so p is similar to an even binary sextic.*

Theorem 5.4. *A binary sextic p has $N(p) = 2$ if and only if p is similar to A_t for $t \in \mathbb{C}$, with the following exceptional values: $N(A_3) = 0$, $N(A_{-1}) = 1$, $N(A_0) = N(A_{15}) = 4$ and $N(A_{-5}) = 6$.*

Theorem 5.5. *A binary sextic p has $N(p) = 3$ if and only if p is similar to B_t for $t \in \mathbb{C}$, except that $N(B_{\pm 2}) = 0$, $N(B_0) = 4$ and $N(B_{\pm 5\sqrt{-2}}) = 6$.*

Theorem 5.6. *The binary sextics p with $N(p) > 3$ are similar to Q_1 or Q_2: $N(Q_1) = 4$ and $N(Q_2) = 6$; Q_1 is similar to A_0, A_{15}, B_0; Q_2 is similar to A_{-5} and $B_{\pm 5\sqrt{-2}}$.*

Proof of Theorem 5.3. Suppose $p = f_1^3 + f_2^3$ is a binary sextic with $N(p) \geq 1$. If f_1 and f_2 are not distinct, then p is a cube, so f_1 and f_2 are distinct. If $\gcd(f_1, f_2) = \ell$ is linear, then $f_1 = \ell\ell_1$ and $f_2 = \ell\ell_2$, where ℓ_1 and ℓ_2 are distinct. Thus, $p = \ell^3(\ell_1^3 + \ell_2^3)$ satisfies (i). If f_1 and f_2 are relatively prime, then by Theorem 2.3, we may make a linear change M so that both $f_1 \circ M$ and $f_2 \circ M$ are even; that is, there exist distinct linear forms ℓ_j so that $(f_j \circ M)(x, y) = \ell_j(x^2, y^2)$; now let $q = \ell_1^3 + \ell_2^3$; this is (ii). □

Theorem 5.7. *If p is a binary sextic with a square factor, then $N(p) \leq 1$.*

Proof. Suppose $\ell^k \mid p$ for a linear factor ℓ, where $k \geq 2$. Suppose $k \geq 3$ and $p = f_1^3 + f_2^3$ for quadratic forms f_1, f_2. Then as in Lemma 2.4, ℓ must divide at least two of $\{f_1 + \omega^k f_2\}$, and so $\ell \mid f_1, f_2$, so p has no other representation as a sum of two cubes.

Now suppose $k = 2$, and after a linear change, take $\ell = y$, so that for some $c_j \in \mathbb{C}$,

$$p(x, y) = \lambda y^2(x + c_1 y)(x + c_2 y)(x + c_3 y)(x + c_4 y).$$

To apply Theorem 5.2, we need to write $p = g_1 g_2 g_3$ for linearly dependent factors. If y divides two of the g_j's, it must divide the third, which is impossible, hence we may assume that $g_1 = y^2$. If $N(p) \ge 2$, then after reindexing if necessary, each of these two different sets is dependent:

$$\{y^2, (x + c_1y)(x + c_2y), (x + c_3y)(x + c_4y)\},$$
$$\{y^2, (x + c_1y)(x + c_3y), (x + c_2y)(x + c_4y)\}.$$

But dependence implies that $c_1 + c_2 = c_3 + c_4$ and $c_1 + c_3 = c_2 + c_4$, so $c_3 = c_2$ and $c_4 = c_1$ and $(x+c_1y)(x+c_2y) = (x+c_3y)(x+c_4y)$, so the factors are not distinct. $\square$

We isolate those exceptional cases in Theorems 5.4 and 5.5 with square factors.

Theorem 5.8. *We have* $N(A_3) = 0$, $N(B_{\pm 2}) = 0$, *and* $N(A_{-1}) = 1$.

Proof. By the first argument of the proof of Theorem 5.7, since $A_3(x, y) = (x^2 + y^2)^3$, in any representation $A_3 = f_1^3 + f_2^3$, both f_1 and f_2 are multiples of $x^2 + y^2$, so that they are not distinct. This also follows from Liouville's solution for Fermat's Last Theorem in polynomials (see [17, pp.263-265] for a proof).

We have seen that if ℓ^2 (but not ℓ^3) divides a sextic p and p has a factorization that partitions into three dependent factors, then one of those factors must be ℓ^2. The only feasible partitions for $B_{\pm 2}(x, y) = (x^3 \pm y^3)^2$ are $\{(x \pm y)^2, (x \pm \omega y)^2, (x \pm \omega^2 y)^2\}$, which are linearly independent; thus $N(B_{\pm 2}) = 0$.

Finally, consider A_{-1}, which factors as $(x - y)^2(x + y)^2(x^2 + y^2)$. Each of the two squares must be a factor, and $\{(x - y)^2, (x + y)^2, x^2 + y^2\} \subset \langle x^2 + y^2, xy\rangle$. There is a representation for $2A_{-1}$ in (3.2) with $\gamma = \sqrt{-4/3}$. Thus $N(A_{-1}) = 1$. $\square$

It is worth mentioning that $A_{-1}(x, y) = (x^2-y^2)^2(x^2+y^2)$, so $A_{-1}(x, y) = q_1(x^2, y^2)$, where $q_1(x, y) = (x-y)^2(x+y)$ is not square-free. But $\tilde{A}_{-1}(x, y) := A_{-1}(x+y, x-y) = 32x^4y^2 + 32x^2y^4 = q_2(x^2, y^2)$, where $q_2(x, y) = 32xy(x + y)$ *is* square-free. Although A_{-1} and $\tilde{A}_{-1}$ are similar, q_1 and q_2 are not.

Now suppose that $N(p) \ge 2$. By Theorem 1.1, we know that after a linear change, p appears as the common sum in (1.10), (1.12) or (1.13), and in the first two cases, $N(p) \ge 3$. Since (1.12) is a linear change of (1.10), we may ignore it. We now apply Theorem 5.1 to $p_{3,\lambda}$ and to $p_{1,\lambda}$, which have already been conveniently split into six linear factors. There are 15 ways to divide six factors into three unordered pairs.

Proof of Theorems 5.4, 5.5 and 5.6. Up to a constant which can be ignored, we have $p_{3,\lambda}(x, y) = xy(x - y)(x + y)(\alpha x + y)(x + \alpha y)$, where $\alpha = \lambda^3 \notin \{0, -1, 1\}$, which cause repeated factors. It is not hard to check the 15 possibilities, and we suppress the details. In two cases, the factors are always dependent:

$$\{x(\alpha x + y), y(x + \alpha y), (x + y)(x - y)\} = \langle \alpha x^2 + xy, xy + \alpha y^2\rangle,$$
$$\{x(x + \alpha y), y(\alpha x + y), (x + y)(x - y)\} = \langle x^2 + \alpha xy, \alpha xy + y^2\rangle.$$

There are two cases when there are multiple dependencies. If $\alpha \in \{\pm 2, \pm\frac{1}{2}\}$, there are two additional cases of dependency, and if $\alpha = \pm i$, there are four additional cases. Thus, $N(p_{3,\lambda}) = 2$ for $\lambda(1 - \lambda^6) \ne 0$ unless $\alpha \in \{\pm 2, \pm\frac{1}{2}, \pm i\}$.

If $\alpha = \lambda^3 = \pm i$, then up to powers of ω, $\lambda^2 = -1$. In the language of Theorem 3.1, $\frac{r_\gamma + s_\gamma}{2} = \lambda^2 \implies r_\gamma + s_\gamma = -2 = (8 + 6\gamma^2)^{1/3} \implies 3(1+\gamma^2) = -5$, so $p_{3,\pm i}$ is similar to A_5. If $\alpha = \pm 2, \pm\frac{1}{2}$, then $\lambda^2 = 2^{\pm 2/3}$ and $r_\gamma + s_\gamma = 2^{1/3}, 2^{5/3} \implies 8 + 6\gamma^2 = 2, 32 \implies 3(1+\gamma^2) = 0, 15$, so $p_{3,\lambda}$ is similar to $A_0 = Q_1$ or A_{15}.

Up to a constant,

$$p_{1,\lambda}(x,y) = (\lambda x + y)(\lambda x + \omega y)(\lambda x + \omega^2 y)(x + \lambda y)(x + \lambda\omega y)(x + \lambda\omega^2 y).$$

As we would hope, there are three cases in which the factors are always dependent:

$$\begin{aligned} &\{(\lambda x + y)(x + \lambda y), (\lambda x + \omega y)(x + \lambda\omega^2 y), (\lambda x + \omega^2 y)(x + \lambda\omega y)\}, \\ &\{(\lambda x + y)(x + \lambda\omega y), (\lambda x + \omega y)(x + \lambda y), (\lambda x + \omega^2 y)(x + \lambda\omega^2 y)\}, \\ &\{(\lambda x + y)(x + \lambda\omega^2 y), (\lambda x + \omega y)(x + \lambda\omega y), (\lambda x + \omega^2 y)(x + \lambda y)\}; \end{aligned} \tag{5.3}$$

the subspaces are $\langle x^2 + \omega^k y^2, xy\rangle$. There are a few cases with multiple dependencies: when $\lambda = \pm i$ (so $\alpha = \pm i$), there is one extra case. In this case, $p_{1,\pm i}(x,y) = \pm 2iQ_1(x,y)$. The other cases in which a dependency occurs are when $\lambda^4 + 4\lambda^2 + 1 = 0$, up to $\lambda \mapsto \omega^k\lambda$. For example, suppose

$$\begin{aligned} &\{(\lambda x + y)(x + \lambda y), (x + \lambda\omega y)(x + \lambda\omega^2 y), (\lambda x + \omega y)(\lambda x + \omega^2 y)\} \\ &= \{\lambda x^2 + (\lambda^2 + 1)xy + \lambda y^2, x^2 - \lambda xy + \lambda y^2, \lambda^2 x^2 - \lambda xy + y^2\} \end{aligned}$$

is linearly dependent. This happens if and only if

$$\begin{vmatrix} \lambda & \lambda^2 + 1 & \lambda \\ 1 & -\lambda & \lambda^2 \\ \lambda^2 & -\lambda & 1 \end{vmatrix} = (\lambda^2 - 1)(\lambda^4 + 4\lambda^2 + 1) = 0.$$

In computations that Ramanujan could probably do in his sleep,

$$\begin{aligned} \lambda^4 + 4\lambda^2 + 1 = 0 \implies \lambda^2 = -2 \pm \sqrt{3} \implies \lambda = \pm_1 \left(\tfrac{\sqrt{6} \pm_2 \sqrt{2}}{2}\right) i \\ \implies \lambda^3 + \lambda^{-3} = \pm 5\sqrt{-2}. \end{aligned} \tag{5.4}$$

Since B_t and B_{-t} are similar via $y \mapsto -y$, we focus on $B_{5\sqrt{-2}}$. Let $\eta = \frac{\sqrt{6}+\sqrt{2}}{2}$, so $\lambda = \eta i$ is a root. We have a linear change with bizarre coefficients:

$$B_{5\sqrt{-2}}(\zeta_8^2 \eta x + \zeta_8 y, x + \zeta_8^3 \eta y) = 54\zeta_8^3\eta^3 Q_2(x,y), \tag{5.5}$$

showing that $B_{5\sqrt{-2}}$ is similar to Q_2. We give a geometric explanation for (5.5) in the next section. □

The instance of (1.6) with the simplest coefficients is probably

$$\begin{aligned} (x^2 + xy - y^2)^3 + (x^2 - xy - y^2)^3 &= 2(x^2)^3 + 2(-y^2)^3 = 2x^6 - 2y^6 \\ &= (\omega x^2 + xy - \omega^2 y^2)^3 + (\omega x^2 - xy - \omega^2 y^2)^3 \\ &= (\omega^2 x^2 + xy - \omega y^2)^3 + (\omega^2 x^2 - xy - \omega y^2)^3. \end{aligned} \tag{5.6}$$

With $(x,y)\mapsto(x+y,x-y)$, (5.6) is due to Girardin in 1910 (see [4, p.550]; the earliest exact version of (5.6) we've found is by Elkies in 1995 (see [3, p.542]). Observe that (5.6) is simply (1.9) with $\lambda=i$ and $y\mapsto iy$, and it also a scaling of Q_1. (We have $2x^6-2y^6=Q_1(rx,sy)$ if $r^6=2, s^6=-2$.) Unsurprisingly, since $\lambda=i$, a flip of (5.6) is similar to Q_2:

$$(\omega x^2+xy-\omega^2y^2)^3-(\omega^2x^2+xy-\omega y^2)^3=-3\sqrt{-3}(x^5y-xy^5). \tag{5.7}$$

Finally, we remark that while (5.6) is presented as a Type(-1) family, we have

$$(x^2+xy-y^2)+(x^2-xy-y^2)=2^{-1/3}(2^{1/3}x^2+(-2^{1/3}y^2)),$$

which gives a Type$(2^{2/3})$ family from (3.2), with $y\mapsto iy$. Thus the Type parameter may vary when more than three representations occur.

6. More on the extra representations

As we saw in the last section, there are two special cases of sextics with more than three representations and we treat them separately. First, note that

$$Q_1(x,y)=x^6+y^6=A_0(x,y)=B_0(x,y);A_{15}(x,y)=\tfrac{1}{2}A_0(x+y,x-y).$$

For purposes of analyzing the factorizations, we note that with $\lambda=i$, it is easier to use powers of $\nu:=\zeta_{12}$:

$$Q_1(x,y)=(x-\nu y)(x-\nu^3y)(x-\nu^5y)(x-\nu^7y)(x-\mu^9y)(x-\nu^{11}y).$$

Keeping in mind that $i=\nu^3,\omega=\nu^4$, and rearranging (5.3) a bit, we have that the three dependent factorizations of Q_1 are:

$$\begin{gathered}\{(x+\nu y)(x+\nu^{11}y),(x+\nu^3y)(x+\nu^9y),(x+\nu^5y)(x+\nu^7y)\},\\ \{(x+\nu y)(x+\nu^3y),(x+\nu^7y)(x+\nu^9y),(x+\nu^5y)(x+\nu^{11}y)\},\\ \{(x+\nu y)(x+\nu^7y),(x+\nu^3y)(x+\nu^5y),(x+\nu^9y)(x+\nu^{11}y)\}.\end{gathered}$$

These live in $\langle x^2+y^2,xy\rangle,\langle x^2+y^2,\omega xy\rangle,\langle x^2+\omega^2y^2,xy\rangle$ respectively. The fourth dependent factorization is

$$\{(x+\nu y)(x+\nu^7y),(x+\nu^3y)(x+\nu^9y),(x+\nu^5y)(x+\nu^{11}y)\}\subseteq\langle x^2,y^2\rangle.$$

The best way of visualizing the four equal pairs of sums seems to be (5.6).

The other case is somewhat more mysterious. Since $Q_2(x,y)=xy(x^4-y^4)$, it is simple to work out all fifteen factorizations into three quadratics. The following six are dependent:

$$\{xy, (x+y)(x+iy), (x-y)(x-iy)\} \subseteq \langle x^2+iy^2, xy\rangle,$$
$$\{xy, (x+y)(x-iy), (x-y)(x+iy)\} \subseteq \langle x^2-iy^2, xy\rangle,$$
$$\{x(x+y), y(x-y), (x+iy)(x-iy)\} \subseteq \langle x^2+xy, x^2+y^2\rangle,$$
$$\{x(x+iy), y(x-iy), (x+y)(x-y)\} \subseteq \langle x^2+ixy, x^2-y^2\rangle,$$
$$\{x(x-y), y(x+y), (x+iy)(x-iy)\} \subseteq \langle x^2-xy, x^2+y^2\rangle,$$
$$\{x(x-iy), y(x+iy), (x+y)(x-y)\} \subseteq \langle x^2-ixy, x^2-y^2\rangle.$$

We could simply write Q_2 explicitly as an element in $\langle F, G\rangle^3$ in these six cases. It is more interesting to derive them from earlier work; see (6.1), (6.3), (6.4) below.

First, observe that $r_{3,i}(x,y) = r_{3,-i}(x,y) = -3\sqrt{-3}\; Q_2(x,y) = (\sqrt{-3})^3 Q_2(x,y)$. One would think that this gives four representations of Q_2, coming from (1.13); however the representation for $\lambda = -i$ is a permutation of that from $\lambda = i$, and there are only two distinct ones:

$$\begin{aligned} -3\sqrt{-3}\; Q_2(x,y) &= (\nu^5x^2+xy+\nu y^2)^3 + (\nu^7x^2-xy+\nu^{11}y^2)^3, \\ 3\sqrt{-3}\; Q_2(x,y) &= (\nu^{11}x^2+xy+\nu^7y^2)^3 + (\nu x^2-xy+\nu^5y^2)^3. \end{aligned} \tag{6.1}$$

These come from $\langle x^2-ixy, x^2-y^2\rangle$ and $\langle x^2+ixy, x^2-y^2\rangle$ respectively. However, $Q_2(x,y) = -iQ_2(x,iy)$, so

$$Q_2(x,y) = f_1(x,y)^3 + f_2(x,y)^3 \implies Q_2(x,y) = (if_1(x,iy))^3 + (if_2(x,iy))^3. \tag{6.2}$$

In this way, we immediately obtain two more representations:

$$\begin{aligned} -3\sqrt{-3}\; Q_2(x,y) &= (\nu^{10}x^2+xy+\nu^8y^2)^3 + (\nu^8x^2-xy+\nu^{10}y^2)^3, \\ -3\sqrt{-3}\; Q_2(x,y) &= (\nu^4x^2+xy+\nu^2y^2)^3 + (\nu^2x^2-xy+\nu^4y^2)^3. \end{aligned} \tag{6.3}$$

These are in $\langle x^2+xy, x^2+y^2\rangle$ and $\langle x^2-xy, x^2+y^2\rangle$, as one would expect; (6.2) simply permutes the equations, and we get no more. Since $\nu^4 = \omega$ and $\nu^2 = -\omega^2$, the second equation in (6.3) recovers (5.7).

We have $Q_2(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}) = Q_2(x,y)$, so after some simplification, we obtain the last two representations of Q_2:

$$\begin{aligned} 6\sqrt{-6}\; Q_2(x,y) &= (\zeta_8^5x^2+\sqrt{6}\;xy+\zeta_8^7y^2)^3 + (\zeta_8x^2+\sqrt{6}\;xy+\zeta_8^3y^2)^3. \\ 6\sqrt{-6}\; Q_2(x,y) &= (\zeta_8^7x^2-\sqrt{-6}\;xy+\zeta_8^5y^2)^3 + (\zeta_8^3x^2-\sqrt{-6}\;xy+\zeta_8y^2)^3. \end{aligned} \tag{6.4}$$

These are in $\langle x^2+iy^2, xy\rangle$ and $\langle x^2-iy^2, xy\rangle$. Although it might seem daunting to consider checking whether any two of these six expressions are similar, the fact that they live in different subspaces shows that this is impossible.

Finally, we discuss the connection of Q_2 and $B_{5\sqrt{-2}}$. To do so, we need an old idea of Felix Klein; see also [15, p.731]. Associate to each non-zero linear form $\ell(x,y) = sx - ty$ the image of $t/s \in \mathbb{C}^*$ on the unit sphere S^2 under the Riemann map and vice-versa. (Assign $\ell(x,y) = y$ to ∞ and $(0,0,1)$.) The *Klein set* of

$p(x,y)=\prod_{j=1}^{k}(s_jx-t_jy)$ is the image of the k points t_j/s_j on S^2 under the Riemann map. Every rotational symmetry of the Klein set of p has an interpretation as a symmetry of p under a linear change.

There are two particularly symmetric six-point sets on S^2. One is a hexagon along a great circle, say the equator. Note that $Q_1(x,y) = x^6+y^6 = \prod_{j=0}^{5}(x+\zeta_{12}^{2j+1}y)$ has such a hexagon as its Klein set. The other natural choice is the vertex set of a regular octahedron, and the Klein set of Q_2 is $\{\pm e_k\}$:

$$Q_2(x,y) = xy(x-y)(x+y)(x-iy)(x+iy) = xy(x^4-y^4).$$

One may rotate an octahedron so that the top and bottom are antipodal triangular faces parallel to the equator. One set of coordinates of the vertices is:

$$\left\{\pm\left(\tfrac{2}{\sqrt{6}},0,\tfrac{\sqrt{2}}{\sqrt{6}}\right),\pm_1\left(\tfrac{-1}{\sqrt{6}},\pm_2\tfrac{\sqrt{3}}{\sqrt{6}},\tfrac{\sqrt{2}}{\sqrt{6}}\right)\right\}. \tag{6.5}$$

The cubic which corresponds to the triangle in the northern hemisphere is

$$(x-\lambda_0y)(x-\omega\lambda_0y)(x-\omega^2\lambda_0y) = x^3-\tfrac{5+3\sqrt{3}}{\sqrt{2}}y^3,\quad \lambda_0=\tfrac{\sqrt{6}+\sqrt{2}}{2}.$$

Similarly, the cubic for the southern hemisphere is

$$(x+\lambda_0^{-1}y)(x+\omega\lambda_0^{-1}y)(x+\omega^2\lambda_0^{-1}y) = x^3+\tfrac{5+3\sqrt{3}}{\sqrt{2}}y^3.$$

Multiplying these together, we get another Klein polynomial for the octahedron:

$$\tilde{Q}_2(x,y) = x^6-5\sqrt{2}\,x^3y^3-y^6 \implies \tilde{Q}_2(x,iy) = x^6+5\sqrt{2}ix^3y^3+y^6 = B_{5\sqrt{-2}}(x,y).$$

The rotation relating $\{\pm e_k\}$ into (6.5) inspired the coefficients of (5.5).

There are, in general, $\frac{(3r)!}{3!(r!)^3}$ ways to arrange the $3r$ linear factors of a form p into three factors of degree r, and by Theorem 5.1, this gives an upper bound on the number of ways to write p as a sum of two cubes. It would be interesting to know how the actual bound grows for p. The natural analogues of Q_1, Q_2 are $x^{3r}+y^{3r}$ and $xy(x^{3r-2}-y^{3r-2})$.

7. Other approaches to sums of two cubes

The proof of the Euler-Binet parameterization of all solutions, found for example in [6, pp.199-201], can easily be adapted to fields of characteristic zero. For our purposes, we look at rational functions over $\mathbb{C}$.

Theorem 7.1 (Euler-Binet). *Suppose $F=\mathbb{C}(x_1,\dots,x_n)$ and suppose*

$$p = f_1^3+f_2^3 = f_3^3+f_4^3. \tag{7.1}$$

for pairwise distinct $f_1,f_2,f_3,f_4\in F$. Then there exist $\mu,a,b\in F$ so that

$$\begin{aligned} f_1 &= \mu(1-(a-3b)(a^2+3b^2)), & f_2 &= \mu((a+3b)(a^2+3b^2)-1),\\ f_3 &= \mu((a+3b)-(a^2+3b^2)^2), & f_4 &= \mu((a^2+3b^2)^2-(a-3b)). \end{aligned} \tag{7.2}$$

Conversely, if f_1,f_2,f_3,f_4 are given by (7.2) in terms of μ,a,b, then

$$p = f_1^3+f_2^3 = f_3^3+f_4^3 = 18\mu^3b(a^2+3b^2)(1-(a+b)^3-(a-b)^3+(a^2+3b^2)^3)). \tag{7.3}$$

Proof. Define g_i's by

$$f_1 = g_1 + g_2, \quad f_2 = g_1 - g_2, \quad f_3 = g_3 + g_4, \quad f_4 = g_3 - g_4, \tag{7.4}$$

so that (7.1) becomes

$$p = 2g_1(g_1^2 + 3g_2^2) = 2g_3(g_3^2 + 3g_4^2).$$

Since $p \neq 0$, $g_1^2 + 3g_2^2 \neq 0$ as well, and we may define

$$a = \frac{g_1g_3 + 3g_2g_4}{g_1^2 + 3g_2^2}, \qquad b = \frac{g_1g_4 - g_3g_2}{g_1^2 + 3g_2^2}. \tag{7.5}$$

Observe that

$$ag_1 - 3bg_2 = g_3; \qquad bg_1 + ag_2 = g_4; \qquad a^2 + 3b^2 = \frac{g_3^2 + 3g_4^2}{g_1^2 + 3g_2^2} = \frac{g_1}{g_3}. \tag{7.6}$$

(In the original derivation, taken over $\mathbb{Q}$, (a, b) are defined by $a \pm b\sqrt{-3} = \frac{g_3 \pm g_4\sqrt{-3}}{g_1 \pm g_2\sqrt{-3}}$, which is unambiguous. We cannot do this here, because some coefficient of g_j might involve $\sqrt{-3}$, but (7.5) recaptures the essence.) Now let

$$\begin{gathered} c = a(a^2 + 3b^2) - 1, \qquad d = 3b(a^2 + 3b^2) \\ \implies cg_1 - dg_2 = (a^2 + 3b^2)(ag_1 - 3bg_2) - g_1 = (a^2 + 3b^2)g_3 - g_1 = 0, \end{gathered} \tag{7.7}$$

so $cg_1 = dg_2$. Suppose $c = d = 0$. Looking at $d = 0$, $a^2 + 3b^2 = 0$ implies $c = -1$, so $b = 0$, and $ag_1 = g_3$, and $ag_2 = g_4$ by (7.6), so that $af_1 = f_3$ and $af_2 = f_4$ implying that (7.1) is not honest. Thus $(c, d) \neq (0, 0)$, and we write (g_1, g_2) with $\mu \in F$ as

$$g_1 = \mu d = 3\mu b(a^2 + 3b^2), \quad g_2 = \mu c = \mu(a(a^2 + 3b^2) - 1). \tag{7.8}$$

Now solve for g_3 and g_4 from (7.6):

$$g_3 = ag_1 - 3bg_2 = 3\mu b, \quad g_4 = bg_1 + ag_2 = \mu((a^2 + 3b^2)^2 - a). \tag{7.9}$$

Plug back in to (7.4) and (7.5) to get (7.2). □

Corollary 7.2. *Suppose f_1, f_2, f_3, f_4 are forms of degree k satisfying* (7.1)*. Then up to a possible common factor, there exist forms p, q, r of degree $\leq 2k$ so that*

$$\begin{gathered} f_1 = r(r^3 - (p - 3q)(p^2 + 3q^2)), \quad f_2 = r((p + 3q)(p^2 + 3q^2) - r^3), \\ f_3 = r^3(p + 3q) - (p^2 + 3q^2)^2, \quad f_4 = (p^2 + 3q^2)^2 - r^3(p - 3q). \end{gathered} \tag{7.10}$$

Proof. Define f_1, f_2, f_3, f_4 as above, and define a and b via (7.5) as rational functions with a common denominator, subject to possible cancellation:

$$a = \frac{p(x, y)}{r(x, y)}, \qquad b = \frac{q(x, y)}{r(x, y)}. \tag{7.11}$$

The expressions for f_3, f_4 have a formal denominator of r^4, so we take $\mu(x, y) = r^4(x, y)$, with the understanding that cancellation may occur. By substituting (7.11) into (7.2), we obtain (7.10). □

Applying this to the quadruple $(f_1, f_2, f_3, f_4) = (F_{6,\lambda}, -F_{4,\lambda}, F_{3,\lambda}, -F_{5,\lambda})$, there is much cancellation and

$$a = -\frac{x^2+y^2}{2\lambda\, xy}, \quad b = \frac{-i(x^2-y^2)}{2\sqrt{3}\lambda\, xy}, \quad \mu = r^4xy, \tag{7.12}$$

so that p and q are quadratic, and r is linear. Other choices for the f_j's lead to p, q, r of higher degree. There are $3^4 \cdot 4! = 1944$ ways to arrange the f_i's, counting cube roots of unity, and we cannot assert that a simpler set of parameters doesn't exist. In the famous Ramanujan case of $12^3 + 1^3 = 10^3 + 9^3 = 1729$, the integral version of (7.2) comes from $(a, b, \mu) = (\frac{10}{19}, \frac{7}{19}, -\frac{361}{42})$, but permuting 9 and 10 means that we need denominators of 266 and 333. On the other hand, the same identity flipped as $10^3 + (-1)^3 = (-9)^3 + 12^3$ comes from $(a, b, \mu) = (-\frac{3}{2}, \frac{1}{2}, 1)$. We note for later use that $5^3 + 4^3 = (-3)^3 + 6^3$ comes from $(a, b, \mu) = (0, 1, \frac{1}{2})$

The other standard approach to equal sums of cubes arises from point-addition on the curve $X^3 + Y^3 = A$; see e.g. [19]. Assuming that $(X, Y) = (X_1, Y_1), (X_2, Y_2)$ lie on this curve, the cubic equation $(tX_1 + (1-t)Y_1)^3 + (tX_2 + (1-t)Y_2)^3 = A$ has two solutions $t = 0, 1$, and so the third may be computed; after simplification,

$$\begin{aligned} X_3 &= \frac{A(X_1 - X_2) + Y_1Y_2(X_2Y_1 - X_1Y_2)}{(X_1^2X_2 + Y_1^2Y_2) - (X_1X_2^2 + Y_1Y_2^2)}, \\ Y_3 &= \frac{A(Y_1 - Y_2) + X_1X_2(X_1Y_2 - X_2Y_1)}{(X_1^2X_2 + Y_1^2Y_2) - (X_1X_2^2 + Y_1Y_2^2)}. \end{aligned} \tag{7.13}$$

This computation (usually done over $\mathbb{Q}$), is still valid when X_i, Y_i are polynomials. Of course, the denominator means that the new solution is usually composed of rational functions. Somewhat astonishingly, (7.13) is applicable to (1.10), and we present a theorem whose only proof is direct computation.

Theorem 7.3. *If we take* $(X_1, Y_1) = (F_{1,\lambda}, F_{2,\lambda})$, $(X_2, Y_2) = (F_{3,\lambda}, F_{4,\lambda})$ *and* $A = p_{1,\lambda}(x, y)$ *in* (7.13)*, then* $(X_3, Y_3) = (F_{5,\lambda}, F_{6,\lambda})$.

More generally, if we take the parameterizations from (7.2) to add (f_1, f_2) and (f_3, f_4), we obtain denominators. But if we add $(f_1, -f_4)$ and $(f_3, -f_2)$, which come from the flip $f_1^3 - f_4^3 = (-f_2)^3 + f_3^3$, we obtain a third polynomial solution which is apparently new.

$$f_1^3 - f_4^3 = -f_2^3 + f_3^3 = (\mu(1 + 2a(a^2 + 3b^2)))^3 - (\mu(2a + (a^2 + 3b^2)^2))^3. \tag{7.14}$$

A few caveats: even though (7.2) is a complete parameterization of solutions to two equal sums of two cubes; (7.14) is *not* a complete parameterization of solutions to three equal sums of two cubes. An extremely tedious application of Theorem 5.1 to the three flips of (7.2) shows that this is the only bonus representation.

As is the case with $\mathbb{Q}$, there can be arbitrarily large sets of equal pairs of sums of two cubes. For example, Rouse and the author give in [16] the complete (infinite) solution to the solution over rational functions of:

$$x^3 + y^3 = \left(\frac{p(x,y)}{r(x,y)}\right)^3 + \left(\frac{q(x,y)}{r(x,y)}\right)^3, \qquad p,q,r \in \mathbb{C}[x,y].$$

for rational functions $(p/r, q/r)$. Clearing the denominator in any finite family of sums $x^3 + y^3 = (\frac{p_i}{r_i})^3 + (\frac{q_i}{r_i})^3, 1 \le i \le N$, gives a set of N equal sums.

We may also take an invariant-theory approach to $N(p) \ge 1$. In any sum of two cubes of quadratic forms:

$$\sum_{j=1}^{2} (\alpha_{j0}x^2 + \alpha_{j1}xy + \alpha_{j2}y^2)^3 = \sum_{k=0}^{6} c_k x^{6-k} y^k,$$

the seven c_k's are cubic polynomials in the six $\alpha'_{j\ell}s$, and since $7 > 6$, we know that the c_k's must be algebraically dependent. There are $\binom{n+6}{6}$ monomials in the c_j's of degree n; these are forms of degree $3n$ in the $\alpha'_{j\ell}s$, which comprise a vector space of dimension $\binom{3n+5}{5}$. Eventually, $\binom{n+6}{6} > \binom{3n+5}{5}$, so there must be dependence at some degree n. Unfortunately, the smallest n for which this happens is $n = 1442$.

We can be less brute-force and apply Theorem 5.1. Suppose our given cubic p is a sum of two cubes, factor it and expand it in the usual way. Write p as

$$\sum_{k=0}^{6} c_k x^{6-k} y^k = c_0 \left(x^6 + \sum_{k=1}^{6} e_k x^{6-k} y^k \right) = c_0 \prod_{j=1}^{6} (x + r_j y),$$

where the e_k's are the elementary symmetric functions in the r_j's. As noted earlier, there are 15 ways to divide the 6 r_j's into 3 pairs of roots, and the condition that the quadratic factors be dependent is equivalent to the vanishing of

$$H(r) := \prod_{\ell=1}^{15} \begin{vmatrix} 1 & 1 & 1 \\ r_{\sigma_\ell(1)} + r_{\sigma_\ell(2)} & r_{\sigma_\ell(3)} + r_{\sigma_\ell(4)} & r_{\sigma_\ell(5)} + r_{\sigma_\ell(6)} \\ r_{\sigma_\ell(1)} r_{\sigma_\ell(2)} & r_{\sigma_\ell(3)} r_{\sigma_\ell(4)} & r_{\sigma_\ell(5)} r_{\sigma_\ell(6)} \end{vmatrix}.$$

where the product is taken over a suitable subset of S_6. (Of course $H(r) = 0$ even if the factors are dependent, so this is a necessary but not sufficient condition.) Mathematica can compute $H(r)$ without too much difficulty, and in a few hours transform it into a symmetric function in the e_k's of degree 15. Now write $e_k = c_k/c_0$, make the substitution and multiply by c_0^{15} to get the relation. It has 1360 terms and is *isobaric* in the old sense: each monomial $\prod c_k^{m_k}$ in the product has $\sum m_k = 15, \sum k m_k = 45$. It seems likely that this is the skew invariant called I_{15} in the old literature. For more information, see [5], especially §143, §244 and Examples 20 and 21 on pp.315-6. The original discovery is attributed there to Joubert.

Finally, here are some of the quadratic parameterizations of (1.6) which can be found in the literature. The earliest one found in [4, p.554] was in J. R. Young's 1816 book *Algebra*, in S. Ward's edition of 1832, and in 1895, by the self-taught mathematician Artemas Martin (see [1]) in a journal he wrote, edited and typeset:

$$\begin{aligned}(x^2+16xy-21y^2)^3+(-x^2+16xy+21y^2)^3+(2x^2-4xy+42y^2)^3\\ =(2x^2+4xy+42y^2)^3.\end{aligned} \tag{7.15}$$

This is a Type(4) family. In fact, Young presented a one-parameter family of such solutions, of Type(n^2), which homogenizes to

$$\begin{aligned}(nx^2-6nxy+3(n^7-n)y^2)^3+(-x^2+6n^3xy+3(n^6-1)y^2)^3\\ =(nx^2+6nxy+3(n^7-n)y^2)^3+(-x^2-6n^3xy+3(n^6-1)y^2)^3.\end{aligned} \tag{7.16}$$

By Theorem 1.1, these are similar to the Narayanan solutions from a century later, and since their sum is an even polynomial, there isn't a third representation.

Hirschhorn has written several papers which explore Ramanujan's approach to (1.6) and related questions. In [7], he conjectured that an "amazing" identity of Ramanujan in his "Lost Notebook" could be proved via the Type(4) identity

$$(x^2+7xy-9y^2)^3+(2x^2-4xy+12y^2)^3=(2x^2+10y^2)^3+(x^2-9xy-y^2)^3, \tag{7.17}$$

and in [8, p.388], he derived this as a special case of a more general formula, which homogenizes to the Type(n^2) identity:

$$\begin{aligned}(3x^2+6n^3xy+(1-n^6)y^2)^3+(3nx^2-6nxy+(n^7-n)y^2)^3\\ =(3x^2-6n^3xy+(1-n^6)y^2)^3+(3nx^2+6nxy+(n^7-n)y^2)^3.\end{aligned} \tag{7.18}$$

Sándor [18] gave a beautiful solution to (1.6) as a conditional polynomial identity. (In 1873, Korneck [4, p.556] (see [18, p.122]) gave a similar family of identities.) Suppose $(w_1,w_2,w_3,w_4)\in\mathbb{C}^4$ satisfy

$$w_1^3+w_2^3=w_3^3+w_4^3, \tag{7.19}$$

and define $h_i=h_i(x,y)$ by

$$\begin{aligned}h_1&=w_1(w_1+w_2)x^2-(w_3^2-w_4^2)xy-w_2(w_3+w_4)y^2,\\ h_2&=w_2(w_1+w_2)x^2+(w_3^2-w_4^2)xy-w_1(w_3+w_4)y^2,\\ h_3&=w_3(w_1+w_2)x^2-(w_1^2-w_2^2)xy-w_4(w_3+w_4)y^2,\\ h_4&=w_4(w_1+w_2)x^2+(w_1^2-w_2^2)xy-w_3(w_3+w_4)y^2.\end{aligned} \tag{7.20}$$

Then

$$h_1^3+h_2^3-h_3^3-h_4^3=(w_1^3+w_2^3-w_3^3-w_4^3)((w_1+w_2)x^2+(w_3+w_4)y^2)^3=0 \tag{7.21}$$

and

$$\frac{h_1+h_2}{h_3+h_4}=\frac{w_1+w_2}{w_3+w_4}; \tag{7.22}$$

that is, $\{h_j(x,y)\}$ forms a Type($\frac{w_1+w_2}{w_3+w_4}$) family.

Observe that in any solution to (1.6), the coefficients of x^2, say (w_i) up to multiple, must satisfy (7.19). S'andor, in [18] divides into two cases, depending on whether

(7.19) has distinct or repeated entries. First assume that the cubes do not pair off. If $(w_i) \subset \mathbb{C}^4$ satisfies (7.19), then for every choice of x, y, $(h_i(x, y))$ will also satisfy (7.19) (Theorem 1). If $w_i, x, y \in \mathbb{Z}$, and (7.20) holds then (7.21) and (7.22) will hold as well (Theorem 2), and conversely, if $w_i, h_i \in \mathbb{Z}^4$ are solutions to (7.19) and satisfy (7.22), then there exist $x, y \in \mathbb{Z}$ so that $h_i = h_i(x, y)$, up to multiple, in the sense of (7.20) (Theorem 3). Theorem 4 states, in effect, that the quotients that may arise in (7.22) are those that can be represented by $a^2 + 3b^2$ over $\mathbb{Q}$. Similar results hold if the (w_i) pair off, either as coefficients of x^2 or y^2 in (7.20) (Theorems 5,6,7).

Finally, we return to the question of which values of T arise for solutions in $\mathbb{Q}[x, y]$. Again, we thank the referee for pointing in this direction.

Theorem 7.4. *In any solution to* (1.6) *with* $f_j \in \mathbb{Q}[x, y]$ *of Type*(T)*, we have* $T = a^2 + 3b^2$ *with* $a, b \in \mathbb{Q}$*. Conversely, if* $T = a^2 + 3b^2$ *with* $a, b \in \mathbb{Q}$*, then there is a solution to* (1.6) *of Type*(T) *with* $f_j \in \mathbb{Q}[x, y]$*.*

Proof. By Theorem 4.3 and (4.4), for any type T family, we have

$$T = \frac{f_3^2 - f_3 f_4 + f_4^2}{f_1^2 - f_1 f_2 + f_2^2} = \frac{(2f_3 - f_4)^2 + 3f_4^2}{(2f_1 - f_2)^2 + 3f_2^2}. \tag{7.23}$$

Suppose $f_j \in \mathbb{Q}[x, y]$. Since $\gcd(f_1, f_2) = 1$, the denominator is never 0, and upon evaluating the right-hand side at $(x, y) = (1, 0)$, say, we have for certain rationals c_j:

$$T = \frac{c_1^2 + 3c_2^2}{c_3^2 + 3c_4^2} = \left(\frac{c_1 c_3 + 3c_2 c_4}{c_3^2 + 3c_4^2}\right)^2 + 3\left(\frac{c_1 c_4 - c_2 c_3}{c_3^2 + 3c_4^2}\right)^2. \tag{7.24}$$

Conversely, suppose $T = a^2 + 3b^2$ with $a, b \in \mathbb{Q}$. Define $f_i \in \mathbb{C}$ by (7.2) with $\mu \neq 0$. Then by (7.4) and (7.6),

$$f_1^3 + f_2^3 = f_3^3 + f_4^3; \qquad \frac{f_1 + f_2}{f_3 + f_4} = \frac{2g_1}{2g_3} = a^2 + 3b^2.$$

Now let $w_i = f_i$ and apply the construction from (7.20) to obtain a Type(T) family with coefficients in $\mathbb{Q}[x, y]$. □

We remark that a (complicated) Type($a^2 + 3b^2$) family on $\mathbb{Q}[x, y]$ also comes from an appropriate scaling of $\mathcal{F}_{2,\sqrt{a^2+3b^2}}$ under the linear change

$$(x, y) \mapsto \left(-\sqrt{a^2 + 3b^2}\, x + \frac{a + \sqrt{-3}\, b}{\sqrt{a^2 + 3b^2}}\, y, \quad (a + \sqrt{-3}\, b)x - y\right).$$

In the simplest non-square case, $T = a^2 + 3b^2 = 3$. Putting $(a, b, \mu) = (0, 1, \frac{1}{2})$ in (7.2) gives $5^3 + 4^3 = (-3)^3 + 6^3$, and putting $(w_1, w_2, w_3, w_4) = (5, 4, -3, 6)$ into (7.20) yields four quadratics, so that (up to a factor of 3),

$$h_1(x, y) = 15x^2 + 9xy - 4y^2, \quad h_2(x, y) = 12x^2 - 9xy - 5y^2,$$
$$h_3(x, y) = -9x^2 - 3xy - 6y^2, \quad h_4(x, y) = 18x^2 + 3xy + 3y^2.$$

Finally, since $h_1^3 + h_2^3 = h_3^3 + h_4^3$ and $h_1 + h_2 = 3(h_3 + h_4)$, the two flips have each have a third representation; namely,

$$h_1^3 - h_4^3 = h_3^3 - h_2^3 = \left(\tfrac{3}{2}x^2 - 18xy + \tfrac{1}{2}y^2\right)^3 - \left(\tfrac{27}{2}x^2 - 6xy + \tfrac{9}{2}y^2\right)^3;$$
$$h_1^3 - h_3^3 = h_4^3 - h_2^3 = \left(\tfrac{216}{13}x^2 + \tfrac{165}{13}xy + \tfrac{72}{13}y^2\right)^3 - \left(\tfrac{102}{13}x^2 + \tfrac{261}{13}xy + \tfrac{34}{13}y^2\right)^3.$$

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