Moment Problems and Inequalities for Power Sums

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When the paper was written, Choi, Lam and I were working on writing psd symmetric forms (especially quartics) as a sum of squares and I was interested in a particular family of examples, although it didn't turn out to be useful for the project.

Suppose one knows that for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\alpha \ge \frac{\left(\sum_{k=1}^n x_k\right)\left(\sum_{k=1}^n x_k^3\right)}{\left(\sum_{k=1}^n x_k^2\right)^2} \ge -\beta.$$

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Then it would follow that both of the symmetric quartic forms

$$\alpha \left(\sum_{k=1}^n x_k^2\right)^2 - \left(\sum_{k=1}^n x_k\right)\left(\sum_{k=1}^n x_k^3\right),$$
$$\beta \left(\sum_{k=1}^n x_k^2\right)^2 + \left(\sum_{k=1}^n x_k\right)\left(\sum_{k=1}^n x_k^3\right)$$

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The same sort of thing is true for the rational function

$$\frac{\left(\sum_{k=1}^n x_k\right)\left(\sum_{k=1}^n x_k^3\right)}{\sum_{k=1}^n x_k^4}.$$

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$$M_r(x) = M_{r,n}(x) = \sum_{k=1}^{n} x_k^r.$$

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There are some standard analytic inequalities for products of power sums, such as the Hölder and Jensen inequalities. They are, in some sense, a generalization of Cauchy-Schwartz, and require the absolute values of the arguments.

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- $M_0(x) = n$.
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- $M_p(|x|) = M_p(x)$ for all x if p is an even integer.

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Multiply the left hand sides and recall that $M_0(|x|) = n$ and $M_4(|x|) = M_4(x)$, so

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If we take x consisting of n 1's, then $M_r(x) = n$ for all r, so the upper bound is clear.
I will show later that the lower bound is more interesting:

$$\frac{M_1(x)M_3(x)}{M_4(x)} \geq -\alpha_n n \quad \text{where} \quad \alpha_n < \frac{1}{8},\ \alpha_n \to \frac{1}{8}.$$

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The real reason the lower bound is not exactly $-\frac{1}{8}$ is that an important parameter $\gamma = 7 + 4\sqrt{3}$ is irrational. It turns out that

$$\alpha_n < -\frac{1}{8} + \frac{1}{1000}\left(\frac{r}{n-r} - \gamma\right)^2, \quad 1 \le r \le n-1.$$

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Note that $7 + 4\sqrt{3} \approx 13.928 \approx \frac{14}{1}$ and $14 + 1 = 15$.
I'll give several proofs of the "1/8" bound, one of which is a simple application of the moment problem.

For other inequalities of this kind, neither the classical inequalities not the moment problem are helpful. The simplest such example is

$$\frac{M_1(x)M_3(x)}{M_2^2(x)}.$$

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$$\max \frac{M_1(x)M_3(x)}{M_2^2(x)} = \frac{3\sqrt{3}}{16}n^{1/2} + \frac{5}{8} + \mathcal{O}(n^{-1/2}),$$

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The extreme values for the inequalities occur when the n-tuple x has $n-1$ 1's and the n-th component t, where t turns out to satisfy a cubic equation; in the second case, this cubic equation has a rational root.

Suppose μ is the measure with unit point masses at $t = x_1, \dots, x_n$. Observe that

$$\frac{M_1(x)M_3(x)}{nM_4(x)} = \frac{(\int_{-\infty}^{\infty} t\ d\mu)(\int_{-\infty}^{\infty} t^3\ d\mu)}{(\int_{-\infty}^{\infty}\ d\mu)(\int_{-\infty}^{\infty} t^4\ d\mu)}.$$

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As is well known, the classical Hamburger moment problem says that, on writing

$$a_j = \int_{-\infty}^{\infty} t^j \, d\mu,$$

for any non-negative measure, the resulting Hankel matrix

$$H := \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$$

is positive semidefinite, and for any choice of a_j's making H psd, there exists a nonnegative measure satisfying

$$a_j = \int_{-\infty}^{\infty} t^j \, d\mu, \quad 0 \le j \le 3; \qquad a_4 \ge \int_{-\infty}^{\infty} t^4 \, d\mu.$$

More generally, the moment problem implies that the power-sum matrix

$$\begin{vmatrix} M_0 & M_1 & \dots & M_n \\ M_1 & M_2 & \dots & M_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ M_n & M_{n+1} & \dots & M_{2n} \end{vmatrix}$$

is always psd; this is a symmetric form of degree $n(n+1)$.

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If μ is the measure with point mass x_j^{2k} at $t = x_j^r$, then the same holds with M_i replaced by M_{2k+ir} but that is another talk.

Let's return to the problem at hand. Since $M_1(x)$ does not divide $M_3(x)$ for $n > 2$, there will be points where they take different signs and so $M_1 M_3 = a_1 a_3 < 0$ at a minimum value. We have

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_2^3 - a_1^2 a_4 - a_3^2 a_0 \geq 0$$

$$\implies a_0 a_2 a_4 - a_2^3 \geq -2a_1 a_2 a_3 - (a_1^2 a_4 + a_3^2 a_0).$$

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Recall that we want to find the smallest value of

$$\frac{a_1 a_3}{a_0 a_4},$$

given that the a_i's satisfy the inequality above. This is a nice undergraduate optimization problem. Let's do it.

Write $a_0 a_4 = \lambda^2$; since we are interested in $a_1 a_3 < 0$, we make the estimate

$$-(a_1^2 a_4 + a_3^2 a_0) \geq -2\sqrt{a_0 a_4 a_1^2 a_3^2} = 2\lambda a_1 a_3,$$

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And so we have

$$\begin{aligned} a_0 a_2 a_4 - a_2^3 &\geq -2a_1 a_2 a_3 - (a_1^2 a_4 + a_3^2 a_0) \geq 0 \\ \implies \lambda^2 a_2 - a_2^3 &\geq -2a_1 a_2 a_3 + 2\lambda a_1 a_3 = 2(-a_1 a_3)(\lambda + a_2) \\ \implies \frac{a_2(\lambda - a_2)}{2} &\geq -a_1 a_3. \end{aligned}$$

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Now we choose $a_2 = \frac{\lambda}{2}$ to see that $\frac{\lambda^2}{8} \geq -a_1 a_3$, which means that

$$\frac{a_1 a_3}{a_0 a_4} = \frac{a_1 a_3}{\lambda^2} \geq -\frac{1}{8}.$$

As is standard in moment problems, the extreme values here occur at measures with two masses. If $x = (r, \dots, r, s, \dots, s)$ with k r's and $n - k$ s's, then the translation of the problem to moments corresponds to a measure with mass k at $x = r$ and $n - k$ at $x = s$. The extreme values above happen when the ratio of the two masses is $7 + 4\sqrt{3}$, which is irrational. I won't go through the details, because this comes up in the other approaches as well.

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A similar computation shows that if a is odd and b are both integers, then

$$\frac{(M_1 M_3)^a (M_2)^{2b}}{(M_0 M_4)^{a+b}} \ge -\frac{1}{2^a}\left(\frac{a}{2a+2b}\right)^a \left(\frac{a+2b}{2a+2b}\right)^{a+2b}.$$

This isn't in the 1983 paper, but I think I'll spare you the very similar proof. Note that $a = 1, b = 0$ recovers $-\frac{1}{8}$.

As I noted in my Tuesday talk, back in the early 80's, Choi, Lam and I were laboring on a long and unpublished manuscript on symmetric quartics, which Charu Goel is now working to edit and complete. One result we had is that a psd symmetric quartic form with a non-trivial zero must have a non-trivial zero with at most two different coordinates. This was also used and presented by William Harris in his 1992 thesis and 1997 paper.

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So in our second method, we look at the values at all such points by considering

$$\frac{(ku+\ell v)(ku^3+\ell v^3)}{(k+\ell)(ku^4+\ell v^4)},$$

over all pairs of integers $k+\ell=n$ and reals (u,v).

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over all pairs of integers $k+\ell=n$ and reals (u,v).

This expression is bihomogeneous in the variables (k,ℓ) and (u,v), and if we let $t=\frac{v}{u}$ (taking $|t|\ge 1$ without loss of generality) and $w=\frac{k}{\ell}$, then this quotient becomes the more tractable

$$F(w,t):=\frac{(w+t)(w+t^3)}{(w+1)(w+t^4)}.$$

There is, ahem, an algebraic identity

$$F(w,t) = \frac{(w+t)(w+t^3)}{(w+1)(w+t^4)} = \frac{9(t^2-w)^2 + w(t^2+4t+1)^2}{8(w+1)(w+t^4)} - \frac{1}{8}.$$

Since $w > 0$, it follows that $F(w,t) \geq -\frac{1}{8}$.

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Since $w > 0$, it follows that $F(w,t) \ge -\frac{1}{8}$.
If $F(w,t) = -\frac{1}{8}$, then $t^2+4t+1=0$ and $w=t^2$; that is, $t = t_0 = -(2+\sqrt{3})$ and $w = t_0^2$, so $\frac{k}{n-k} = t_0^2 = \gamma = 7+4\sqrt{3}$.
This cannot happen.

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$$F(w,t) = \frac{(w+t)(w+t^3)}{(w+1)(w+t^4)} = \frac{9(t^2-w)^2 + w(t^2+4t+1)^2}{8(w+1)(w+t^4)} - \frac{1}{8}.$$

Since $w > 0$, it follows that $F(w,t) \geq -\frac{1}{8}$.
If $F(w,t) = -\frac{1}{8}$, then $t^2 + 4t + 1 = 0$ and $w = t^2$; that is, $t = t_0 = -(2+\sqrt{3})$ and $w = t_0^2$, so $\frac{k}{n-k} = t_0^2 = \gamma = 7 + 4\sqrt{3}$.
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It is not hard to show that as $n \to \infty$, γ can be approximated arbitrarily well by fractions of the shape $w_n = \frac{k_n}{n-k_n}$. Assuming $w_n > 13$, we find that $-\frac{1}{8} + \frac{1}{1000}(w_n - \gamma)^2 > \alpha_n > -\frac{1}{8}$, and α_n gets squeezed to $-\frac{1}{8}$ as $n \to \infty$. I could go into the Diophantine approximation issues but that's another talk.

A third proof. By the first two, the symmetric quartic form $8M_1M_3 + nM_4$ is positive definite for all n. It's obviously then sos for $n = 2, 3$. What about larger n? (If it's sos, then this also provides a proof of psd.) Thirty-five years ago, a proof or refutation of sos-ness seemed beyond the bounds of our hand-computation.

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I found this by what used to be called "the method of judicious guessing". By setting all summands equal to zero, it's not hard to find that no zero of the left hand side exists because, you guessed it, $7 + 4\sqrt{3} \notin \mathbb{Q}$. Once again, we see that $8M_1M_3 + nM_4$ is a positive definite n-ary quartic for all n.

There is an established literature of looking at the extreme points of the power-sum map, going back at least to Ursell in the 50s. The following is a toy version.

Theorem

Let $F(u, v, w)$ be a differentiable function and suppose the extreme values of $G(x) = F(M_1(x), M_2(x), M_3(x))$ occur at a point $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$. Then either there are at most two different values for the $\bar{x}_j$'s, or F itself is singular at $(M_1(\bar{x}), M_2(\bar{x}), M_3(\bar{x}))$

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Proof.

At an extreme point $\bar{x}$, for each j, we have

$$0 = \frac{\partial G}{\partial x_j}(\bar{x}) = \frac{\partial G}{\partial u}(\bar{x}) + \frac{\partial G}{\partial v}(\bar{x})(2x_j) + \frac{\partial G}{\partial w}(\bar{x})(3x_j^2)$$

and so either the x_j's satisfy a quadratic, or all coefficients are zero. □

Let's quickly look at $\frac{M_1 M_3}{M_2^2}$. It is clear that uw/v^2 has no singular points at which one would find a non-zero max or min. Suppose x has $k \geq \frac{n}{2}$ components equal to u and $n-k$ equal to v. Then evaluation at $\bar{x}$ gives:

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Once again, letting $t = \frac{v}{u}$ and $w = \frac{k}{n-k}$ leads to a computation of the extreme values of

$$\frac{(w+t)(w+t^3)}{(w+t^2)^2}.$$

For fixed w, calculus leads to the cubic $t^3 - 3t^2 - 3tw + w = 0$, which has three real roots, and

$$t = 1 + 2\sqrt{w+1}\cos\theta; \qquad \cos 3\theta = \frac{1}{\sqrt{w+1}}.$$

I'll skip the details here, but plugging back in, one finds that the largest positive and negative values occur when w is as large as possible (that is $w = \frac{n-1}{1} = n-1$), and the announced asymptotics follow routinely.

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A quick and dirty way to find the asymptotics is to take an n-tuple x consisting of $n-1$ 1's and then one component equal to $u\sqrt{n-1}$. Then $M_r = (n-1) + u^r(n-1)^{r/2}$ and we have

$$\frac{M_1 M_3}{M_2^2} = \frac{((n-1) + u\sqrt{n-1})((n-1) + u^3(n-1)^{3/2}))}{(1+u^2)^2(n-1)^2}.$$

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By taking the biggest terms in the numerator, we find an approximate value of

$$\frac{(n-1) * u^3(n-1)^{3/2}}{(1+u^2)^2(n-1)^2} = \frac{u^3}{(u^2+1)^2} \cdot \sqrt{n-1}.$$

Calculus shows that the maximum and minimum of the given function of u occur at $u = \pm\sqrt{3}$ and give the values $\pm\frac{3\sqrt{3}}{16}$.

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It is not hard to show that for $n \geq 2$ and $a_0 = a_4 = n, a_1 = a_3 = 1$, $a_2 = \frac{3}{n}$, the matrix

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$$

is positive definite and we have $a_1 a_3 / a_2^2 = n^2/9$ which is not globally bounded, not even if we put in the asymptotic factor of $a_0^{1/2}$ in the denominator.

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What this means for moments is that the moment quotient

$$\frac{M_1 M_3}{M_0^{1/2} M_2^2} = \frac{a_1 a_3}{a_0^{1/2} a_2^2}$$

is unbounded over all non-negative measures with $a_0 = n$, though it is bounded (essentially by $\pm\frac{3\sqrt{3}}{16}$) over measures consisting of n equal point masses. Serious question: does this phenomenon merit more attention?

Some final thoughts.

- There is a general theorem in the paper for ratios of products of power sums. Suppose all parameters are non-negative integers and $\sum_j a_j p_j = bq$. Then there are easily computable constants c_i so that

$$\max \frac{M_{p_1}^{a_1} \cdots M_{p_n}^{a_n}}{M_q^b} = c_1 n^{c_2} + \mathcal{O}(n^{c_3}).$$

  There are many cases in which

$$\min \frac{M_{p_1}^{a_1} \cdots M_{p_n}^{a_n}}{M_q^b} = c_4 n^{c_2} + o(n^{c_2}),$$

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although c_4 may be hard to find.
- I can report no progress in computing λ_n so that $M_2^2 + \lambda_n M_1 M_3$ is sos.
- There are a lot more other inequalities lurking about. For example, $\inf \frac{M_1(x)M_5(x)}{nM_6(x)}$ seems to be $-\frac{1}{4}$.

About that reboot.

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I'm thinking Jennifer Lawrence as the Hankel matrix and a special guest appearance by Arnold Schwarzenegger as the Nullstellensatz.

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$$\int_\Omega f_u \, dt.$$