Analytic Number Theory Seminar Notes, UIUC, 9/21/00
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(These notes contain all formulas from the 9/19/00 overheads, and a bit more and are part of the draft of "Equal sums of higher powers of quadratic forms". The paper will contain proofs, references, thanks and detailed historical remarks).

We are interested in finding all non-trivial solutions to the equation

$$p = f_1^m + f_2^m = f_3^m + f_4^m, \qquad m \geq 3 \tag{1}$$

in quadratic forms $f_j = f_j(x,y) \in \mathbf{C}[x,y]$. By "non-trivial", we mean that $p \neq 0$, that $\{f_1^m, f_2^m\} \neq \{f_3^m, f_4^m\}$, and that there do not exist $\alpha_j \in \mathbf{C}$ and $g \in \mathbf{C}[x,y]$ so that $\alpha_1^m + \alpha_2^m = \alpha_3^m + \alpha_4^m$ and $f_j = \alpha_j g$. If (1) holds for $\{f_j\}$ and $g_j(x,y) = f_j(r_1x + s_1y, r_2x + s_2y)$ for some invertible linear change of variables, then (1) holds for $\{g_j\}$ as well. We will consider two instances of (1) to be the same if one is obtained from the other in this way, and either will be called an *avatar* of the underlying family.

There is an extensive literature on solutions to (1) over $\mathbf{Z}[x,y]$, usually construed as a parameterization to the solutions to (1) over $\mathbf{Z}$. Many of our solutions have already appeared in the literature; however, we have been able to find a one-parameter family of solutions to $p = f_1^3 + f_2^3 = f_3^3 + f_4^3 = f_5^3 + f_6^3$ over $\mathbf{Z}[x,y]$ which appears to be new, and which contains all solutions to (1) for $m = 3$. Ramanujan used the identity

$$\begin{aligned}(3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3 \\ = (6x^2 - 4xy + 4y^2)^3.\end{aligned} \tag{2}$$

There are three ways to transpose (2) into the shape of (1). Ramanujan (apparently) did not use the fact that two of them give a third representation:

$$\begin{aligned}&(4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3 \\ &= (6x^2 - 4xy + 4y^2)^3 - (3x^2 + 5xy - 5y^2)^3 \\ &= (6x^2 - 8xy + 6y^2)^3 - (3x^2 - 11xy + 3y^2)^3,\end{aligned} \tag{3}$$

and

$$\begin{aligned}&(4x^2 - 4xy + 6y^2)^3 + (3x^2 + 5xy - 5y^2)^3 \\ &= (6x^2 - 4xy + 4y^2)^3 - (5x^2 - 5xy - 3y^2)^3 \\ &= \left(\tfrac{94}{21}x^2 - \tfrac{8}{21}xy + \tfrac{94}{21}y^2\right)^3 + \left(\tfrac{23}{21}x^2 - \tfrac{199}{21}xy + \tfrac{23}{21}y^2\right)^3.\end{aligned} \tag{4}$$

We note (3) becomes (4), up to the order of the summands, under the unimodular linear transformation

$$(x,y) \to \left(\frac{5x - 2y}{\sqrt{21}}, \frac{3x + 3y}{\sqrt{21}}\right), \tag{5}$$

so that they are avatars of the same identity. The other transposed version of (2),

$$\begin{aligned}&(3x^2+5xy-5y^2)^3+(5x^2-5xy-3y^2)^3\\&=(6x^2-4xy+4y^2)^3+(-4x^2+4xy-6y^2)^3,\end{aligned}\tag{6}$$

does not have a third representation as a sum of two cubes of quadratics.

Our plan of attack is simple: we first show that if (1) holds non-trivially, then f_1 and f_2 must be relatively prime. It follows that there is an invertible linear change of variables which simultaneously diagonalizes f_1 and f_2, so that they (and hence p) are even forms.

Suppose more generally that

$$q(x,y)=(ax^2+bxy+cy^2)^m+(dx^2+exy+fy^2)^m\tag{7}$$

is an even polynomial. If $b=e=0$, then we shall call (7) a *flat* representation of q; otherwise, it is *skew*. Two kinds of skew representations exist for arbitrarily large m:

$$\begin{aligned}&(ax^2+bxy+cy^2)^m+(ax^2-bxy+cy^2)^m,\\&(ax^2+cy^2)^m+(exy)^m\quad\text{for even } m.\end{aligned}\tag{8}$$

We call these *tame skew* representations. Any others, should they exist, would not be "formally" even; we call them *wild skew* representations. It turns out that wild skew representations exist only for $m=3,4,5$. If (7) is wild skew, then taking $y\to -y$ gives a second skew representation for q.

Using Sylvester's algorithm on sums of m-th powers of linear forms, we show that a given even form of degree $2m$ has at most one flat representation. Thus, any solution to (1), after an invertible linear change of variables, consists of a flat and a skew representation of p. We conduct an exhaustive search for all possible wild skew representations. Finally, we use Sylvester's algorithm to determine which even forms with a skew representation also have a flat one.

We now present a census of the solutions to (1). In each case, we have sought the simplest avatars of the solutions; in some cases, p is not even and in some cases, we present an *ad hoc* derivation. We will often bring factors out of the polynomials rather than put their m-th root in: this is a luxury in working over $\mathbf{C}$, rather than $\mathbf{Z}$, since our definition does not distinguish f_j and ϵf_j in (1), where $\epsilon^m=1$.

The solutions to (1) for $m=3$ all flow from the following identity for $\alpha\in\mathbf{C}$:

$$\begin{aligned}&(\alpha^2-1)(\alpha x^3+y^3)(x^3+\alpha y^3)\\&=(\alpha x^2-xy+\alpha y^2)^3+\alpha(-x^2+\alpha xy-y^2)^3.\end{aligned}\tag{9}$$

This can be checked by writing the second expression in (9) as

$$\begin{aligned}&\alpha^3(x^2+y^2)^3-3\alpha^2(x^2+y^2)^2(xy)+3\alpha(x^2+y^2)(xy)^2-(xy)^3\\&-\alpha(x^2+y^2)^3+3\alpha^2(x^2+y^2)^2(xy)-3\alpha^3(x^2+y^2)(xy)^2+\alpha^4(xy)^3\\&=(\alpha^3-\alpha)\left((x^2+y^2)^3-3(x^2+y^2)(xy)^2\right)+(\alpha^4-1)(xy)^3\\&=(\alpha^3-\alpha)(x^6+y^6)+(\alpha^4-1)(xy)^3\end{aligned}\tag{10}$$

Letting $\omega = e^{2\pi i/3}$ be a primitive cube root of unity, and taking $(x,y) \to (\omega^2 x, \omega y)$, $(\omega x, \omega^2 y)$, we see that (9) yields a three-fold solution to (1):

$$\begin{aligned}(\alpha^2-1)(\alpha x^3+y^3)(x^3+\alpha y^3) &= \\ = (\alpha x^2 - xy + \alpha y^2)^3 &+ \alpha(-x^2+\alpha xy - y^2)^3 \\ = (\omega\alpha x^2 - xy + \alpha\omega^2 y^2)^3 &+ \alpha(-\omega x^2 + \alpha xy - \omega^2 y^2)^3 \\ = (\omega^2\alpha x^2 - xy + \alpha\omega y^2)^3 &+ \alpha(-\omega^2 x^2 + \alpha xy - \omega y^2)^3.\end{aligned} \tag{11}$$

If α is real, then the forms in (11) are $\mathbf{Q}$-linear combinations of x^2+y^2, $\sqrt{-3}(x^2-y^2)$ and xy. Thus, we make the linear change $(x,y) \to (x-\sqrt{-3}\ y, x+\sqrt{-3}\ y)$, write $\alpha = \lambda^3$ and bring α inside the second cubes, and define

$$\begin{aligned}F_1(x,y;\lambda) &= (2\lambda^3-1)x^2 - 3(2\lambda^3+1)y^2; \\ F_2(x,y;\lambda) &= (\lambda^4-2\lambda)x^2 + 3(\lambda^4+2\lambda)y^2; \\ F_3(x,y;\lambda) &= -(1+\lambda^3)x^2 - 6\lambda^3 xy + 3(\lambda^3-1)y^2; \\ F_4(x,y;\lambda) &= (\lambda^4+\lambda)x^2 + 6\lambda xy + 3(\lambda^4-\lambda)y^2; \\ F_5(x,y;\lambda) &= -(1+\lambda^3)x^2 + 6\lambda^3 xy + 3(\lambda^3-1)y^2; \\ F_6(x,y;\lambda) &= (\lambda^4+\lambda)x^2 - 6\lambda xy + 3(\lambda^4-\lambda)y^2.\end{aligned} \tag{12}$$

Then

$$\begin{aligned}(\lambda^6-1)\big((1+\lambda^3)^2x^6 &+ 9(1-10\lambda^3+\lambda^6)x^4y^2 \\ +27(1+10\lambda^3+\lambda^6)x^4y^2 &+ 27(1-\lambda^3)^2y^6\big) \\ &= F_1^3(x,y;\lambda)+F_2^3(x,y;\lambda) \\ &= F_3^3(x,y;\lambda)+F_4^3(x,y;\lambda) \\ &= F_5^3(x,y;\lambda)+F_6^3(x,y;\lambda).\end{aligned} \tag{13}$$

For example, if $\lambda = 2$, then (13) implies

$$\begin{aligned}(15x^2-51y^2)^3 &+ (12x^2+60y^2)^3 \\ = (-9x^2-48xy+21y^2)^3 &+ (18x^2+12xy+42y^2)^3 \\ = (-9x^2+48xy+21y^2)^3 &+ (18x^2-12xy+42y^2)^3.\end{aligned} \tag{14}$$

After making the linear change $(x,y) \to (2x-y, y)$ in (14), we obtain (2) up to the multiplicative constant 12^3; similarly, after making the linear change $(x,y) \to (8x+y, 2x-5y)$, we obtain (3), up to the multiplicative constant 252^3.

A second family of solutions to (1) with $m=3$ follows from (8) and the fact that a general binary cubic is a sum of two cubes of linear forms. For $\mu \in \mathbf{C}$, the skew representation $p(x,y) = (x^2+\mu xy+y^2)^3+(x^2-\mu xy+y^2)^3 = 2(x^2+y^2)^3+6\mu^2x^2y^2(x^2+$

y^2) "should" also have a flat representation of the form $(c_1x^2+c_2y^2)^3+(c_3x^2+c_4y^2)^3$. Indeed, after taking $y \to iy$, writing $\mu = (\frac{4i\nu}{\nu^2+3})^{1/2}$ (ironically, to avoid square roots), and clearing the denominator, we obtain

$$\begin{aligned}
2(\nu^4-9)(x^2-y^2)\left((\nu^2+3)^2x^4-(\nu^4-18\nu^2+9)x^2y^2+(\nu^2+3)^2y^4\right) \\
= (\nu^2+3)\big((\nu^2+2\nu-3)x^2-(\nu^2-2\nu-3)y^2\big)^3 \\
+(\nu^2+3)\big((\nu^2-2\nu-3)x^2-(\nu^2+2\nu-3)y^2\big)^3 \\
= (\nu^2-3)\big((\nu^2+3)x^2+4\nu xy-(\nu^2+3)y^2\big)^3 \\
+(\nu^2-3)\big((\nu^2+3)x^2-4\nu xy-(\nu^2+3)y^2\big)^3.
\end{aligned} \tag{15}$$

It is far from obvious, but upon setting

$$(x,y) \to (-3(1-\nu)x+3(1+\nu)y, -(\nu+3)x+(3-\nu)y), \qquad \lambda \to \left(\frac{\nu^2-3}{\nu^2+3}\right)^{1/3},$$

in (13), and dividing by $-72\nu(\nu^2+3)^{-4/3}$, we find that

$$\begin{aligned}
F_1 &\to (\nu^2+3)^{1/3}\left((\nu^2+2\nu-3)x^2-(\nu^2-2\nu-3)y^2\right), \\
F_2 &\to -(\nu^2-3)^{1/3}\left((\nu^2+3)x^2-4\nu xy-(\nu^2+3)y^2\right), \\
F_3 &\to (\nu^2+3)^{1/3}\left((\nu^2-2\nu-3)x^2-(\nu^2+2\nu-3)y^2\right), \\
F_4 &\to (\nu^2+3)^{1/3}\left((\nu^2+3)x^2+4\nu xy-(\nu^2+3)y^2\right),
\end{aligned} \tag{16}$$

Thus, (15) is an avatar for $F_1^3+(-F_3)^3 = F_4^3+(-F_2)^3$. However, (13) does not automatically imply a third expression of this kind, and none exists. In (2), $\lambda = 2$, hence $\nu = \sqrt{-27/7}$, under which (15) gives (6) after a linear change. By our definition, (13) and (15) are different solutions to (1), but they are clearly related.

There are two cases for $m = 3$ in which (13) and (15) coalesce. The first is

$$\begin{aligned}
2x^6-2y^6 &= 2(x^2)^3+2(-y^2)^3 \\
&= (x^2+xy-y^2)^3+(x^2-xy-y^2)^3 \\
&= (x^2+\omega xy-\omega^2y^2)^3+(x^2-\omega xy-\omega^2y^2)^3 \\
&= (x^2+\omega^2xy-\omega y^2)^3+(x^2-\omega^2xy-\omega y^2)^3.
\end{aligned} \tag{17}$$

The second involves the innocuous-looking sextic $p(x,y) = x^6-5x^4y^2-5x^2y^4+y^6$, which has six representations as a sum of two cubes of quadratic forms. These are most easily seen by taking a different avatar: $p(x+y, i(x-y)) = 32x^5y+32xy^5$. If we then let $\eta = \frac{\sqrt{6}+\sqrt{2}}{4} + i \cdot \frac{\sqrt{6}-\sqrt{2}}{4} = e^{\pi i/12}$, we have

$$\begin{aligned} q(x,y) = x^5y + xy^5 &= \frac{1}{6\sqrt{6}}\left((x^2+\sqrt{6}xy-y^2)^3 + (-x^2+\sqrt{6}xy+y^2)^3\right) \\ &= \frac{1}{6\sqrt{6}}\left((ix^2-\sqrt{6}xy+iy^2)^3 + (-ix^2-\sqrt{6}xy-iy^2)^3\right) \\ &= \frac{1}{3\sqrt{3}}\left((\eta x^2+xy+\eta^{11}y^2)^3 + (\eta^5x^2-xy+\eta^7y^2)^3\right) \\ &= \frac{1}{3\sqrt{3}}\left((\eta^{11}x^2+xy+\eta y^2)^3 + (\eta^7x^2-xy+\eta^5y^2)^3\right) \\ &= \frac{1}{3\sqrt{3}}\left((-\eta x^2+xy-\eta^{11}y^2)^3 + (-\eta^5x^2-xy-\eta^7y^2)^3\right) \\ &= \frac{1}{3\sqrt{3}}\left((-\eta^{11}x^2+xy-\eta y^2)^3 + (-\eta^7x^2-xy-\eta^5y^2)^3\right). \end{aligned} \tag{18}$$

The first two and last four representations in (18) are linked by the symmetries of q: $q(x,y) = q(y,x) = q(ix,-iy)$.

It is extremely difficult to use (13) and (15) directly in order to check for these coalescences, since each of these formulas must be considered with changes of variable as well. Fortunately, a more algebraic approach comes to the rescue.

It is well-known that there is a simple criterion for a polynomial over **C** (of any degree, in any number of variables) to be a sum of two squares of polynomials: that it be reducible. Indeed, if $p = f^2 + g^2$, then p $= (f+ig)(f-ig)$, and if $p = FG$, then $p = \left(\frac{F+G}{2}\right)^2 + \left(\frac{F-G}{2i}\right)^2$.

It does not appear to be widely known that there is a simple criterion for a polynomial over **C** (of any degree, in any number of variables) to be a sum of two cubes of polynomials: *it is either already a cube, or can be written as a product of three linearly dependent, but non-proportional, factors.* Suppose $p = f^3 + g^3$. If f and g are proportional, then p is itself a cube. Othwerwise, we have

$$\begin{aligned} p = f^3 + g^3 = (f+g)(f+\omega g)(f+\omega^2 g); \\ 1\cdot(f+g) + \omega\cdot(f+\omega g) + \omega^2\cdot(f+\omega^2 g) = 0. \end{aligned} \tag{19}$$

(Note that if $\omega^j \neq \omega^k$, then f and g are proportional if and only if $f + \omega^j g$ and $f+\omega^k g$ are proportional.) Conversely, suppose $p = FGH$, where $H = aF + bG$, with $ab \neq 0$. Then the following identity is easily checked

$$p = FG(aF+bG) = \frac{1}{3(\omega^2-\omega)ab}\left((aF-\omega bG)^3 - (aF-\omega^2 bG)^3\right). \tag{20}$$

The way we use this theorem is as follows: suppose p is even and has a flat representation as a sum of two cubes. After some preliminaries, we may assume that

$$p(x,y) = (x^2-a^2y^2)(x^2-b^2y^2)(x^2-c^2y^2), \tag{21}$$

where a^2, b^2, c^2 are distinct and non-zero. There are fifteen different ways to pair up the linear factors $x \pm ay, x \pm by, x \pm cy$ into three quadratics as $p = FGH$. The three even quadratics in (21) are always dependent, and this gives the flat representation. In each of the other cases, linear dependence of the factors imposes an algebraic relation on a, b, c. Up to sign and permutation, (15) corresponds to $ac = b^2$, and (13) corresponds to $(a+b)(a+c)(b+c) = 8abc$. We then examine which values of (a, b, c) solve more than one of these relations. We might have used this approach to solve (1) for $m = 3$ directly, but it does not seem to be practical for $m \geq 4$.

When $m = 4$, there are three solutions to (1), up to change of variable, and one of them is three-fold. The first is immediate from the properties of roots of unity and most easily expressed in transposed form:

$$(x^2 + y^2)^4 + (\omega x^2 + \omega^2 y^2)^4 + (\omega^2 x^2 + \omega y^2)^4 = 18(xy)^4, \tag{21}$$

Although (21) is complex, if we make the substitution $(x, y) \to (x + \sqrt{-3}\, y, x - \sqrt{-3}\, y)$, we get a solution over $\mathbf{Z}$:

$$(2x^2 - 6y^2)^4 + (x^2 + 6xy - 3y^2)^4 + (x^2 - 6xy - 3y^2)^4 = 18(x^2 + 3y^2)^4. \tag{22}$$

Another avatar of (21) (take $(x, y) \to (x - \omega y), i(x - \omega^2 y))$ is, up to multiple,

$$(x^2 + 2xy)^4 + (2xy + y^2)^4 + (x^2 - y^2)^4 = 2(x^2 + xy + y^2)^4 \tag{23}$$

The second solution for $m = 4$ can be derived by supposing that there is a solution to (1) of the shape $(f+g)^4 + (f-g)^4 = (f+h)^4 + (f-h)^4$ with $g \neq \pm h$. This relation implies that $6f^2 + g^2 + h^2 = 0$, and this can be parameterized in the usual Pythagorean triple manner, by taking $f = \sqrt{2}xy$, $g = \sqrt{3}(x^2 - y^2)$ and $h = i\sqrt{3}\,(x^2 + y^2)$, to give

$$\begin{aligned} &18x^8 - 28x^4y^4 + 18y^8 \\ &= (\sqrt{3}\, x^2 + \sqrt{2}\, xy - \sqrt{3}\, y^2)^4 + (\sqrt{3}\, x^2 - \sqrt{2}\, xy - \sqrt{3}\, y^2)^4 \\ &= (\sqrt{3}\, x^2 + i\sqrt{2}\, xy + \sqrt{3}\, y^2)^4 + (\sqrt{3}\, x^2 - i\sqrt{2}\, xy + \sqrt{3}\, y^2)^4 \end{aligned} \tag{24}$$

This expression can almost be put over $\mathbf{Z}$ by taking $y \to \sqrt{6}\, y$ and dividing through by 9, but there will still be "i"'s in the second sum. There is no avatar of (24) over $\mathbf{R}$, let alone $\mathbf{Z}$.

The third solution for $m = 4$ is three-fold, and also has no avatar over $\mathbf{R}$:

$$\begin{aligned} H(x, y) &= \big(8\sqrt{3}\big)xy(x^6 - y^6) = \\ &(x^2 + \sqrt{3}xy - y^2)^4 - (x^2 - \sqrt{3}xy - y^2)^4 \\ &= (\omega^2 x^2 + \sqrt{3}xy - \omega y^2)^4 - (\omega^2 x^2 - \sqrt{3}xy - \omega y^2)^4 \\ &= (\omega x^2 + \sqrt{3}xy - \omega^2 y^2)^4 - (\omega x^2 - \sqrt{3}xy - \omega^2 y^2)^4. \end{aligned} \tag{25}$$

The last two representations follow from the first, since $H(x,y) = H(\omega x, \omega^2 y) = H(\omega^2 x, \omega y)$.

Finally, if $m = 5$, there is a single solution, which can be derived by supposing a solution to (1) of the form $(f+g)^5 + (f-g)^5 = (f+h)^5 + (f-h)^5$ with $f \neq 0, g \neq \pm h$. This relation implies immediately that $2f^2 + g^2 + h^2 = 0$, and again, can be parameterized in the usual Pythagorean triple manner, by taking $f = \sqrt{2}xy$, $g = x^2 - y^2$ and $h = i(x^2 + y^2)$ to give:

$$\begin{aligned} &2\sqrt{2}xy(5x^8 - 6x^4y^4 + 5y^8) \\ &= \left(x^2 + \sqrt{2}xy - y^2\right)^5 + \left(-x^2 + \sqrt{2}xy + y^2\right)^5 \\ &= (ix^2 + \sqrt{2}xy + iy^2)^5 + (-ix^2 + \sqrt{2}xy + -iy^2)^5 \end{aligned} \tag{26}$$

As with (24), the second expression can be made integral by setting $y \to \sqrt{2}y$, however, the second is still complex, and we shall show that there is no real avatar. The four quadratic forms in (26) are special in that, up to linear change, they are the *only* four quadratics whose fifth powers are non-trivially dependent. One would hope that they display some nice symmetry. In fact, if we factor them as $\gamma_k(x - \alpha_k y)(x - \beta_k y)$ for complex numbers $\alpha_k, \beta_k, \gamma_k,\ k = 1,2,3,4$, then the four pairs (α_k, β_k) are the images under stereographic projection of the antipodal pairs of the cube with vertices $(\pm\sqrt{2/3}, 0, \pm\sqrt{1/3}), (0, \pm\sqrt{2/3}, \pm\sqrt{1/3})$.

There are no solutions to (1) for $m \geq 6$.

Here's a bit on the Narayanan family of solutions. In 1913, Ramanujan posed to the *Journal of the Indian Mathematical Society* the question of verifying (2), and "find other quadratic expressions satisfying similar relations". S. Narayanan gave the more general expression:

$$\begin{aligned} (px^2 + mxy - my^2)^3 + (nx^2 - nxy + \ell y^2)^3 + (mx^2 - mxy - py^2)^3 \\ = (\ell x^2 - nxy + ny^2)^3 \end{aligned} \tag{27}$$

where

$$\ell = \lambda(\lambda^3 + 1), \quad m = 2\lambda^3 - 1, \quad n = \lambda(\lambda^3 - 2), \quad p = \lambda^3 + 1 \tag{28}$$

It is not hard to derive (28) by mimicking the pattern of (2) and replacing 3,4,5,6 with the parameters p, n, m, ℓ. One would expect that (27) imposes seven equations on these parameters, but in fact there are only two. It is also not hard to derive (27) from $F_1^3 + F_2^3 = F_3^3 + F_4^3$ in (13): after making the substitution $(x,y) \to \frac{1}{2}(2x - y, y)$, and in the notation of (28), and keeping λ fixed, we get the complete picture:

$$\begin{aligned} &(mx^2 - mxy - py^2)^3 + (nx^2 - nxy + \ell y^2)^3 \\ &= (\ell x^2 - nxy + n^4y^2)^3 - (px^2 + mxy - my^2)^3 \\ &= (\ell x^2 + (n - 2\ell)xy + \ell y^2)^3 - (px^2 - (m + 2p)xy + py^2)^3 \end{aligned} \tag{29}$$

Putting $\lambda = 2$, we have $(\ell, m, n, p, m+2p, n-2\ell) = (18, 15, 12, 9, 33, -24)$, and after dividing by 3, (29) becomes (3). There is a general analogue of (4) as well:

$$\begin{aligned}(px^2 + mxy - my^2)^3 + (nx^2 - nxy + \ell y^2)^3 \\ = (\ell x^2 - nxy + ny^2)^3 - (mx^2 - mxy - py^2)^3 \\ = \frac{1}{(\lambda^6 - 1)^3}\left((rx^2 - sxy + ry^2)^3 + (tx^2 - uxy + ty^2)^3\right)\end{aligned} \tag{30}$$

where

$$\begin{aligned} r &= \lambda^9 - 5\lambda^6 + 2\lambda^3 - 1, \qquad & s &= 4\lambda^9 - 5\lambda^6 + 8\lambda^3 - 1, \\ t &= \lambda^{10} - 2\lambda^7 + 5\lambda^4 - \lambda, \qquad & u &= \lambda^{10} - 8\lambda^7 + 5\lambda^4 - 4\lambda. \end{aligned} \tag{31}$$

Finally, a statement of Sylvester's theorem on linear forms. Two linear forms $\alpha x + \beta y$ and $\alpha' x + \beta' y$ are called *distinct* if they are pairwise linearly independent. Suppose $\{\alpha_j x + \beta_j y : 1 \le j \le r\}$ is a set of pairwise distinct linear forms and suppose

$$p(x, y) = \sum_{k=0}^{m} \binom{m}{k} a_k x^{m-k} y^k.$$

Then there exist $\lambda_j \in \mathbf{C}$ so that

$$p(x, y) = \sum_{j=1}^{r} \lambda_j (\alpha_j x + \beta_j y)^m$$

if and only if

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-r} & a_{m-r+1} & \cdots & a_m \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where

$$q(u, v) = \sum_{\ell=0}^{r} c_\ell u^{r-\ell} v^\ell = \prod_{j=1}^{r} (\beta_j u - \alpha_j v).$$

For example, let $p(x, y) = 2x^3 + 9x^2y + 15x^2y + 9y^3$. Then $(a_0, a_1, a_2, a_3) = (2, 3, 5, 9)$ and with $r = 2$, the equation

$$\begin{pmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $2u^2 - 3uv + v^2 = (2u - v)(u - v)$ mean that there exist λ_k so that $p(x, y) = \lambda_1 (x + y)^3 + \lambda_2 (x + 2y)^3$. In fact $\lambda_k \equiv 1$.