Equal sums of two cubes

Bruce Reznick
University of Illinois at Urbana-Champaign

The Intersection of Number Theory and Combinatorics
AMS Central Sectional Meeting, El Paso, TX, Sept. 18, 2022

One of the most famous anecdotes in all of mathematics comes from G. H. Hardy, about Srinivasa Ramanujan:

One of the most famous anecdotes in all of mathematics comes from G. H. Hardy, about Srinivasa Ramanujan:

> *I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."*

One of the most famous anecdotes in all of mathematics comes from G. H. Hardy, about Srinivasa Ramanujan:

> *I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."*

One of the most famous anecdotes in all of mathematics comes from G. H. Hardy, about Srinivasa Ramanujan:

> *I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."*

Indeed,

One of the most famous anecdotes in all of mathematics comes from G. H. Hardy, about Srinivasa Ramanujan:

> *I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."*

Indeed,

$$1729 = 1728 + 1 = 1000 + 729 = 12^3 + 1^3 = 10^3 + 9^3.$$

One of the most famous anecdotes in all of mathematics comes from G. H. Hardy, about Srinivasa Ramanujan:

> *I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."*

Indeed,

$$1729 = 1728 + 1 = 1000 + 729 = 12^3 + 1^3 = 10^3 + 9^3.$$

If Hardy had come in taxi cab number 91, Ramanujan could have said that it is the smallest positive integer that is a sum of two cubes of positive or negative integers in two different ways:

One of the most famous anecdotes in all of mathematics comes from G. H. Hardy, about Srinivasa Ramanujan:

> *I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."*

Indeed,

$$1729 = 1728 + 1 = 1000 + 729 = 12^3 + 1^3 = 10^3 + 9^3.$$

If Hardy had come in taxi cab number 91, Ramanujan could have said that it is the smallest positive integer that is a sum of two cubes of positive or negative integers in two different ways:

$$91 = 27 + 64 = 216 - 125 = 3^3 + 4^3 = 6^3 + (-5)^3.$$

More than twenty years ago, Bruce Berndt presented the following to the Illinois number theory seminar:

More than twenty years ago, Bruce Berndt presented the following to the Illinois number theory seminar:

In 1913, Ramanujan posed to the *Journal of the Indian Mathematical Society* the following question: "Shew that

$$(6x^2 - 4xy + 4y^2)^3 = (3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3,$$

and find other quadratic expressions satisfying similar relations."

More than twenty years ago, Bruce Berndt presented the following to the Illinois number theory seminar:

In 1913, Ramanujan posed to the *Journal of the Indian Mathematical Society* the following question: "Shew that

$$(6x^2 - 4xy + 4y^2)^3 = (3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3,$$

and find other quadratic expressions satisfying similar relations."
By setting $(x, y) = (1, 0)$ in there, you get $6^3 = 3^3 + 4^3 + 5^3$.

More than twenty years ago, Bruce Berndt presented the following to the Illinois number theory seminar:

In 1913, Ramanujan posed to the *Journal of the Indian Mathematical Society* the following question: "Shew that

$$(6x^2 - 4xy + 4y^2)^3 = (3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3,$$

and find other quadratic expressions satisfying similar relations."
By setting $(x, y) = (1, 0)$ in there, you get $6^3 = 3^3 + 4^3 + 5^3$.
With $(x, y) = (1, -2)$, you get

$$30^3 = (-27)^3 + 36^3 + 3^3$$

More than twenty years ago, Bruce Berndt presented the following to the Illinois number theory seminar:

In 1913, Ramanujan posed to the *Journal of the Indian Mathematical Society* the following question: "Shew that

$$(6x^2 - 4xy + 4y^2)^3 = (3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3,$$

and find other quadratic expressions satisfying similar relations."

By setting $(x, y) = (1, 0)$ in there, you get $6^3 = 3^3 + 4^3 + 5^3$.
With $(x, y) = (1, -2)$, you get

$$30^3 = (-27)^3 + 36^3 + 3^3 \iff 10^3 = (-9)^3 + 12^3 + 1^3.$$

Ramanujan's question about a generalization seems approachable by treating 3, 4, 5, 6 as variables. Consider the proposed:

Ramanujan's question about a generalization seems approachable by treating 3, 4, 5, 6 as variables. Consider the proposed:

$$(\ell x^2 - nxy + ny^2)^3 = (px^2 + mxy - my^2)^3 + (nx^2 - nxy + \ell y^2)^3 + (mx^2 - mxy - py^2)^3.$$

Ramanujan's question about a generalization seems approachable by treating 3, 4, 5, 6 as variables. Consider the proposed:

$$(\ell x^2 - nxy + ny^2)^3 = (px^2 + mxy - my^2)^3 + (nx^2 - nxy + \ell y^2)^3 + (mx^2 - mxy - py^2)^3.$$

There are ostensibly seven equations for the coefficients of the monomials $x^k y^{6-k}$, but these reduce to only two:

Ramanujan's question about a generalization seems approachable by treating 3, 4, 5, 6 as variables. Consider the proposed:

$$(\ell x^2 - nxy + ny^2)^3 = (px^2 + mxy - my^2)^3 + (nx^2 - nxy + \ell y^2)^3 + (mx^2 - mxy - py^2)^3.$$

There are ostensibly seven equations for the coefficients of the monomials $x^k y^{6-k}$, but these reduce to only two:

$$m^3 + n^3 = \ell^3 + p^3 = \ell^2 n + p^2 m.$$

Ramanujan's question about a generalization seems approachable by treating 3, 4, 5, 6 as variables. Consider the proposed:

$$(\ell x^2 - nxy + ny^2)^3 = (px^2 + mxy - my^2)^3 + (nx^2 - nxy + \ell y^2)^3 + (mx^2 - mxy - py^2)^3.$$

There are ostensibly seven equations for the coefficients of the monomials $x^k y^{6-k}$, but these reduce to only two:

$$m^3 + n^3 = \ell^3 + p^3 = \ell^2 n + p^2 m.$$

The solution can be parameterized, and my guess is that this is how S. Narayanan found the explicit solution in 1914:

Ramanujan's question about a generalization seems approachable by treating 3, 4, 5, 6 as variables. Consider the proposed:

$$(\ell x^2 - nxy + ny^2)^3 = (px^2 + mxy - my^2)^3 + (nx^2 - nxy + \ell y^2)^3 + (mx^2 - mxy - py^2)^3.$$

There are ostensibly seven equations for the coefficients of the monomials $x^k y^{6-k}$, but these reduce to only two:

$$m^3 + n^3 = \ell^3 + p^3 = \ell^2 n + p^2 m.$$

The solution can be parameterized, and my guess is that this is how S. Narayanan found the explicit solution in 1914:

$$\ell = c\lambda(\lambda^3 + 1),\ m = c(2\lambda^3 - 1),\ n = c\lambda(\lambda^3 - 2),\ p = c(\lambda^3 + 1).$$

Ramanujan's question about a generalization seems approachable by treating 3, 4, 5, 6 as variables. Consider the proposed:

$$(\ell x^2 - nxy + ny^2)^3 = (px^2 + mxy - my^2)^3 + (nx^2 - nxy + \ell y^2)^3 + (mx^2 - mxy - py^2)^3.$$

There are ostensibly seven equations for the coefficients of the monomials $x^k y^{6-k}$, but these reduce to only two:

$$m^3 + n^3 = \ell^3 + p^3 = \ell^2 n + p^2 m.$$

The solution can be parameterized, and my guess is that this is how S. Narayanan found the explicit solution in 1914:

$$\ell = c\lambda(\lambda^3 + 1),\ m = c(2\lambda^3 - 1),\ n = c\lambda(\lambda^3 - 2),\ p = c(\lambda^3 + 1).$$

Put $\lambda = 2, c = \frac{1}{3}$ to get Ramanujan's original identity.

Everything I am going to talk about today is based on a paper I wrote for the Bruce Berndt 80th Birthday celebration in 2019: "Equal sums of two cubes of quadratic forms", *Int. J. Number Theory*, 17 (2021) 761-786. You can find it on the arXiv or my web page.

Everything I am going to talk about today is based on a paper I wrote for the Bruce Berndt 80th Birthday celebration in 2019: "Equal sums of two cubes of quadratic forms", *Int. J. Number Theory*, 17 (2021) 761-786. You can find it on the arXiv or my web page.

The study of equal sums of cubes of integers goes back hundreds of years; the first quadratic parameterization is from 1816 (see Dickson's *History*, vol. II).

Everything I am going to talk about today is based on a paper I wrote for the Bruce Berndt 80th Birthday celebration in 2019: "Equal sums of two cubes of quadratic forms", *Int. J. Number Theory*, 17 (2021) 761-786. You can find it on the arXiv or my web page.

The study of equal sums of cubes of integers goes back hundreds of years; the first quadratic parameterization is from 1816 (see Dickson's *History*, vol. II).

There are two standard approaches to equal sums of two cubes, which I'll mention briefly: elliptic curves and the Euler-Binet parameterization. The novelty here is viewing the parameterized $a^3 + b^3 = c^3 + d^3$ as a single Diophantine equation, but in $\mathbb{C}[x, y]$.

Everything I am going to talk about today is based on a paper I wrote for the Bruce Berndt 80th Birthday celebration in 2019: "Equal sums of two cubes of quadratic forms", *Int. J. Number Theory*, 17 (2021) 761-786. You can find it on the arXiv or my web page.

The study of equal sums of cubes of integers goes back hundreds of years; the first quadratic parameterization is from 1816 (see Dickson's *History*, vol. II).

There are two standard approaches to equal sums of two cubes, which I'll mention briefly: elliptic curves and the Euler-Binet parameterization. The novelty here is viewing the parameterized $a^3 + b^3 = c^3 + d^3$ as a single Diophantine equation, but in $\mathbb{C}[x, y]$.

One important and utterly trivial point is that each identity has many equivalent expressions, and it will make a difference; e.g.,

Everything I am going to talk about today is based on a paper I wrote for the Bruce Berndt 80th Birthday celebration in 2019: "Equal sums of two cubes of quadratic forms", *Int. J. Number Theory*, 17 (2021) 761-786. You can find it on the arXiv or my web page.

The study of equal sums of cubes of integers goes back hundreds of years; the first quadratic parameterization is from 1816 (see Dickson's *History*, vol. II).

There are two standard approaches to equal sums of two cubes, which I'll mention briefly: elliptic curves and the Euler-Binet parameterization. The novelty here is viewing the parameterized $a^3 + b^3 = c^3 + d^3$ as a single Diophantine equation, but in $\mathbb{C}[x, y]$.

One important and utterly trivial point is that each identity has many equivalent expressions, and it will make a difference; e.g.,

$$a^3 + b^3 = c^3 + d^3 \iff a^3 + b^3 = d^3 + c^3 \iff$$
$$a^3 + (-c)^3 = (-b)^3 + d^3 \iff a^3 + (-d)^3 = (-b)^3 + c^3, \quad \text{etc..}$$

There is of course a massive history of work on the equation $a^3 + b^3 = c^3 + d^3$ over $\mathbb{Q}$, and many known parameterizations and I can't do it full justice here. What's new is the abundance of three-fold solutions.

There is of course a massive history of work on the equation $a^3 + b^3 = c^3 + d^3$ over $\mathbb{Q}$, and many known parameterizations and I can't do it full justice here. What's new is the abundance of three-fold solutions.

The first standard method is "point-addition" on an elliptic curve, using the fact that if you know that a cubic in t has roots at $t = 0$ and $t = 1$, then you can find the third root easily:

There is of course a massive history of work on the equation $a^3 + b^3 = c^3 + d^3$ over $\mathbb{Q}$, and many known parameterizations and I can't do it full justice here. What's new is the abundance of three-fold solutions.

The first standard method is "point-addition" on an elliptic curve, using the fact that if you know that a cubic in t has roots at $t = 0$ and $t = 1$, then you can find the third root easily:

$$f_1^3 + f_2^3 = g_1^3 + g_2^3 = p \implies h_1^3 + h_2^3 = p, \quad \text{where}$$
$$h_1 = f_1 + t(g_1 - f_1),\ h_2 = f_2 + t(g_2 - f_2),$$

There is of course a massive history of work on the equation $a^3 + b^3 = c^3 + d^3$ over $\mathbb{Q}$, and many known parameterizations and I can't do it full justice here. What's new is the abundance of three-fold solutions.

The first standard method is "point-addition" on an elliptic curve, using the fact that if you know that a cubic in t has roots at $t = 0$ and $t = 1$, then you can find the third root easily:

$$f_1^3 + f_2^3 = g_1^3 + g_2^3 = p \implies h_1^3 + h_2^3 = p, \quad \text{where}$$
$$h_1 = f_1 + t(g_1 - f_1),\ h_2 = f_2 + t(g_2 - f_2),$$
$$t = \frac{f_1^2 g_1 + f_2^2 g_2 - f_1^3 - f_2^3}{f_1^2 g_1 - f_1 g_1^2 + f_2^2 g_2 - f_2 g_2^2}.$$

There is of course a massive history of work on the equation $a^3 + b^3 = c^3 + d^3$ over $\mathbb{Q}$, and many known parameterizations and I can't do it full justice here. What's new is the abundance of three-fold solutions.

The first standard method is "point-addition" on an elliptic curve, using the fact that if you know that a cubic in t has roots at $t = 0$ and $t = 1$, then you can find the third root easily:

$$f_1^3 + f_2^3 = g_1^3 + g_2^3 = p \implies h_1^3 + h_2^3 = p, \quad \text{where}$$
$$h_1 = f_1 + t(g_1 - f_1),\ h_2 = f_2 + t(g_2 - f_2),$$
$$t = \frac{f_1^2 g_1 + f_2^2 g_2 - f_1^3 - f_2^3}{f_1^2 g_1 - f_1 g_1^2 + f_2^2 g_2 - f_2 g_2^2}.$$

For example, the points $(12, 1)$ and $(10, 9)$ combine to give $(\frac{46}{3}, -\frac{37}{3})$. But the order matters: $(12, 1)$ and $(9, 10)$ combine to give $(\frac{453}{26}, -\frac{397}{26})$. One can write iterative formulas, which become rational functions of increasingly large degrees.

This second result is in Hardy & Wright.

This second result is in Hardy & Wright.

Theorem (Euler-Binet)

If $a^3 + b^3 = c^3 + d^3$ *for* $a, b, c, d \in \mathbb{Q}$*, then there exist* $r, s, t \in \mathbb{Q}$ *so that*

$$\begin{aligned} a &= (1 - (r - 3s)(r^2 + 3s^2))t, \\ b &= ((r + 3s)(r^2 + 3s^2) - 1)t, \\ c &= (r + 3s - (r^2 + 3s^2)^2)t, \\ d &= ((r^2 + 3s^2)^2 - (r - 3s))t. \end{aligned}$$

This second result is in Hardy & Wright.

Theorem (Euler-Binet)

If $a^3 + b^3 = c^3 + d^3$ for $a, b, c, d \in \mathbb{Q}$, then there exist $r, s, t \in \mathbb{Q}$ so that

$$a = (1 - (r - 3s)(r^2 + 3s^2))t,$$
$$b = ((r + 3s)(r^2 + 3s^2) - 1)t,$$
$$c = (r + 3s - (r^2 + 3s^2)^2)t,$$
$$d = ((r^2 + 3s^2)^2 - (r - 3s))t.$$

So, $1^3 + 12^3 = 10^3 + 9^3$ comes from $(r, s, t) = \left(\frac{107}{266}, -\frac{111}{266}, -\frac{2527}{333}\right)$,

This second result is in Hardy & Wright.

Theorem (Euler-Binet)

If $a^3 + b^3 = c^3 + d^3$ for $a, b, c, d \in \mathbb{Q}$, then there exist $r, s, t \in \mathbb{Q}$ so that

$$\begin{aligned} a &= (1 - (r - 3s)(r^2 + 3s^2))t, \\ b &= ((r + 3s)(r^2 + 3s^2) - 1)t, \\ c &= (r + 3s - (r^2 + 3s^2)^2)t, \\ d &= ((r^2 + 3s^2)^2 - (r - 3s))t. \end{aligned}$$

So, $1^3 + 12^3 = 10^3 + 9^3$ comes from $(r, s, t) = \left(\frac{107}{266}, -\frac{111}{266}, -\frac{2527}{333}\right)$, but $10^3 + (-1)^3 = (-9)^3 + 12^3$ comes from $(r, s, t) = \left(-\frac{3}{2}, \frac{1}{2}, 1\right)$.

Only minor modifications are needed to show that the Euler-Binet parameterization also applies when $\mathbb{Q}$ is replaced by $\mathbb{C}(x_1, \dots, x_n)$; the reliance on $\sqrt{-3} \notin \mathbb{Q}$ can be dealt with formally. Degree considerations show that one can't hope to have r, s, t be polynomials in order to obtain quadratic solutions of our kind.

Only minor modifications are needed to show that the Euler-Binet parameterization also applies when $\mathbb{Q}$ is replaced by $\mathbb{C}(x_1, \dots, x_n)$; the reliance on $\sqrt{-3} \notin \mathbb{Q}$ can be dealt with formally. Degree considerations show that one can't hope to have r, s, t be polynomials in order to obtain quadratic solutions of our kind.

These actually work together: we can combine a flipped version of Euler-Binet with elliptic-curve point-addition:

Only minor modifications are needed to show that the Euler-Binet parameterization also applies when $\mathbb{Q}$ is replaced by $\mathbb{C}(x_1, \dots, x_n)$; the reliance on $\sqrt{-3} \notin \mathbb{Q}$ can be dealt with formally. Degree considerations show that one can't hope to have r, s, t be polynomials in order to obtain quadratic solutions of our kind.

These actually work together: we can combine a flipped version of Euler-Binet with elliptic-curve point-addition:

$$a^3 + (-d)^3 = (-b)^3 + c^3 = e^3 + f^3,$$

where

$$e = (2r(r^2 + 3s^2) + 1)t, \quad f = -(2r + (r^2 + 3s^2)^2)t$$

Only minor modifications are needed to show that the Euler-Binet parameterization also applies when $\mathbb{Q}$ is replaced by $\mathbb{C}(x_1, \dots, x_n)$; the reliance on $\sqrt{-3} \notin \mathbb{Q}$ can be dealt with formally. Degree considerations show that one can't hope to have r, s, t be polynomials in order to obtain quadratic solutions of our kind.

These actually work together: we can combine a flipped version of Euler-Binet with elliptic-curve point-addition:

$$a^3 + (-d)^3 = (-b)^3 + c^3 = e^3 + f^3,$$

where

$$e = (2r(r^2 + 3s^2) + 1)t, \quad f = -(2r + (r^2 + 3s^2)^2)t$$

and where (e, f) is the "elliptic curve sum" of $(a, -d)$ and $(-b, c)$. Caution: it is not true that $(a, -d, -b, c, e, f)$ give a *complete* parameterization of three equal sums of pairs of cubes.

Only minor modifications are needed to show that the Euler-Binet parameterization also applies when $\mathbb{Q}$ is replaced by $\mathbb{C}(x_1, \ldots, x_n)$; the reliance on $\sqrt{-3} \notin \mathbb{Q}$ can be dealt with formally. Degree considerations show that one can't hope to have r, s, t be polynomials in order to obtain quadratic solutions of our kind.

These actually work together: we can combine a flipped version of Euler-Binet with elliptic-curve point-addition:

$$a^3 + (-d)^3 = (-b)^3 + c^3 = e^3 + f^3,$$

where

$$e = (2r(r^2 + 3s^2) + 1)t, \quad f = -(2r + (r^2 + 3s^2)^2)t$$

and where (e, f) is the "elliptic curve sum" of $(a, -d)$ and $(-b, c)$. Caution: it is not true that $(a, -d, -b, c, e, f)$ give a *complete* parameterization of three equal sums of pairs of cubes.

An appeal to the method herein shows that this is the only "bonus" polynomial representation from Euler-Binet.

The utility of this approach is that we find that Ramanujan's identity can be rewritten as two equal sums of two cubes in three different ways. There will be some surprises.

The utility of this approach is that we find that Ramanujan's identity can be rewritten as two equal sums of two cubes in three different ways. There will be some surprises.

First we have

$$(4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3$$
$$= (6x^2 - 4xy + 4y^2)^3 - (3x^2 + 5xy - 5y^2)^3$$

The utility of this approach is that we find that Ramanujan's identity can be rewritten as two equal sums of two cubes in three different ways. There will be some surprises.

First we have

$$
\begin{aligned}
&(4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3 \\
&= (6x^2 - 4xy + 4y^2)^3 - (3x^2 + 5xy - 5y^2)^3 \\
&= (6x^2 - 8xy + 6y^2)^3 - (3x^2 - 11xy + 3y^2)^3.
\end{aligned}
$$

The utility of this approach is that we find that Ramanujan's identity can be rewritten as two equal sums of two cubes in three different ways. There will be some surprises.

First we have

$$\begin{aligned}&(4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3\\ &= (6x^2 - 4xy + 4y^2)^3 - (3x^2 + 5xy - 5y^2)^3\\ &= (6x^2 - 8xy + 6y^2)^3 - (3x^2 - 11xy + 3y^2)^3.\end{aligned}$$

Let me note that this isn't a singular accident. My main result is that if $q_1^3 + q_2^3 = q_3^3 + q_4^3$ for binary quadratic q_j, then of course $q_1^3 + (-q_4)^3 = q_3^3 + (-q_2)^3$ and $q_1^3 + (-q_3)^3 = q_4^3 + (-q_2)^3$. In two of these three cases, there is a *third* representation of the sum as a sum of two cubes of quadratic forms.

A different transposition also has a third representation:

$$(6x^2 - 4xy + 4y^2)^3 - (5x^2 - 5xy - 3y^2)^3$$
$$= (4x^2 - 4xy + 6y^2)^3 + (3x^2 + 5xy - 5y^2)^3$$

A different transposition also has a third representation:

$$
\begin{aligned}
&(6x^2 - 4xy + 4y^2)^3 - (5x^2 - 5xy - 3y^2)^3 \\
&= (4x^2 - 4xy + 6y^2)^3 + (3x^2 + 5xy - 5y^2)^3 \\
&= \left(\tfrac{94}{21}x^2 - \tfrac{8}{21}xy + \tfrac{94}{21}y^2\right)^3 + \left(\tfrac{23}{21}x^2 - \tfrac{199}{21}xy + \tfrac{23}{21}y^2\right)^3 .
\end{aligned}
$$

A different transposition also has a third representation:

$$\begin{aligned}&(6x^2 - 4xy + 4y^2)^3 - (5x^2 - 5xy - 3y^2)^3\\ &= (4x^2 - 4xy + 6y^2)^3 + (3x^2 + 5xy - 5y^2)^3\\ &= \left(\tfrac{94}{21}x^2 - \tfrac{8}{21}xy + \tfrac{94}{21}y^2\right)^3 + \left(\tfrac{23}{21}x^2 - \tfrac{199}{21}xy + \tfrac{23}{21}y^2\right)^3.\end{aligned}$$

Furthermore, this second set of identities can be derived from the first by making a unimodular linear change of variables:

$$(x, y) \to \left(\frac{5x - 2y}{\sqrt{21}}, \frac{3x + 3y}{\sqrt{21}}\right).$$

A different transposition also has a third representation:

$$\begin{aligned}&(6x^2-4xy+4y^2)^3-(5x^2-5xy-3y^2)^3\\&=(4x^2-4xy+6y^2)^3+(3x^2+5xy-5y^2)^3\\&=\left(\tfrac{94}{21}x^2-\tfrac{8}{21}xy+\tfrac{94}{21}y^2\right)^3+\left(\tfrac{23}{21}x^2-\tfrac{199}{21}xy+\tfrac{23}{21}y^2\right)^3.\end{aligned}$$

Furthermore, this second set of identities can be derived from the first by making a unimodular linear change of variables:

$$(x,y)\to\left(\frac{5x-2y}{\sqrt{21}},\frac{3x+3y}{\sqrt{21}}\right).$$

Alas, the third transposition can be proven not to have a third representation as a sum of two cubes of quadratics.

It turns out that these properties (of a third representation, and the equivalence under linear change), are not specific to Ramanujan's example. One can also write down comparable versions for the Narayanan formulas.

It turns out that these properties (of a third representation, and the equivalence under linear change), are not specific to Ramanujan's example. One can also write down comparable versions for the Narayanan formulas.

More to the point, up to changes of variable and taking $\lambda \in \mathbb{C}$, it turns out that (up to flipping terms from one side to another and stray powers of the cube root of unity), the Narayanan formula *completely describes* the solution in binary quadratic forms to

$$q_1^3 + q_2^3 = q_3^3 + q_4^3.$$

It turns out that these properties (of a third representation, and the equivalence under linear change), are not specific to Ramanujan's example. One can also write down comparable versions for the Narayanan formulas.

More to the point, up to changes of variable and taking $\lambda \in \mathbb{C}$, it turns out that (up to flipping terms from one side to another and stray powers of the cube root of unity), the Narayanan formula *completely describes* the solution in binary quadratic forms to

$$q_1^3 + q_2^3 = q_3^3 + q_4^3.$$

As remarked earlier, our analysis comes from looking at the equation in quadratic forms over $\mathbb{C}$ rather than over $\mathbb{Q}$.

The simplest quadratic form identity over $\mathbb{C}$ is a really beautiful one which seems to have been largely overlooked. (With a linear change, it's by Girardin (1910); as given, the earliest I know is Elkies, 1995). It's not so applicable to make equal sums of cubes over $\mathbb{Q}$, because of the "$(\pm 2)^{1/3}$" on the right hand side:

The simplest quadratic form identity over $\mathbb{C}$ is a really beautiful one which seems to have been largely overlooked. (With a linear change, it's by Girardin (1910); as given, the earliest I know is Elkies, 1995). It's not so applicable to make equal sums of cubes over $\mathbb{Q}$, because of the "$(\pm 2)^{1/3}$" on the right hand side:

$$(x^2 + xy - y^2)^3 + (x^2 - xy - y^2)^3 = 2(x^2)^3 - 2(y^2)^3.$$

The simplest quadratic form identity over $\mathbb{C}$ is a really beautiful one which seems to have been largely overlooked. (With a linear change, it's by Girardin (1910); as given, the earliest I know is Elkies, 1995). It's not so applicable to make equal sums of cubes over $\mathbb{Q}$, because of the "$(\pm 2)^{1/3}$" on the right hand side:

$$(x^2 + xy - y^2)^3 + (x^2 - xy - y^2)^3 = 2(x^2)^3 - 2(y^2)^3.$$

Since the right hand side is a polynomial in (x^3, y^3), we can get additional new pairs of summands on the left hand side by sending $y \mapsto \omega y$ and $y \mapsto \omega^2 y$, where ω is a primitive cube root of 1, so we obtain *four* pairs of cubes of complex binary quadratic forms which each sum to $2x^6 - 2y^6$.

The simplest quadratic form identity over $\mathbb{C}$ is a really beautiful one which seems to have been largely overlooked. (With a linear change, it's by Girardin (1910); as given, the earliest I know is Elkies, 1995). It's not so applicable to make equal sums of cubes over $\mathbb{Q}$, because of the "$(\pm 2)^{1/3}$" on the right hand side:

$$(x^2 + xy - y^2)^3 + (x^2 - xy - y^2)^3 = 2(x^2)^3 - 2(y^2)^3.$$

Since the right hand side is a polynomial in (x^3, y^3), we can get additional new pairs of summands on the left hand side by sending $y \mapsto \omega y$ and $y \mapsto \omega^2 y$, where ω is a primitive cube root of 1, so we obtain *four* pairs of cubes of complex binary quadratic forms which each sum to $2x^6 - 2y^6$.

We don't have time to talk about it here, but, the record number of different representations of a binary as a sum of two cubes of quadratics is *six*, held by $xy(x^4 - y^4)$, up to changes of variable.

There is one master identity:

There is one master identity:

$$\begin{aligned}&(\alpha x^2 - xy + \alpha y^2)^3 + \alpha(-x^2 + \alpha xy - y^2)^3 \\ &= (\alpha^2 - 1)(\alpha x^3 + y^3)(x^3 + \alpha y^3) := p_\alpha(x, y).\end{aligned} \tag{1}$$

There is one master identity:

$$(\alpha x^2 - xy + \alpha y^2)^3 + \alpha(-x^2 + \alpha xy - y^2)^3 \\ = (\alpha^2 - 1)(\alpha x^3 + y^3)(x^3 + \alpha y^3) := p_\alpha(x, y). \qquad (1)$$

This is "non-trivial" as long as $\alpha \notin \{0, \pm 1\}$. (The Girardin/Elkies formula can be derived from taking $\alpha = i$ and $y \mapsto iy$.) Note that the factor of α is on the outside; one can put $\alpha^{1/3}$ inside, and also note that p_α is a polynomial in x^3 and y^3.

There is one master identity:

$$
\begin{aligned}
&(\alpha x^2 - xy + \alpha y^2)^3 + \alpha(-x^2 + \alpha xy - y^2)^3 \\
&= (\alpha^2 - 1)(\alpha x^3 + y^3)(x^3 + \alpha y^3) := p_\alpha(x, y).
\end{aligned} \tag{1}
$$

This is "non-trivial" as long as $\alpha \notin \{0, \pm 1\}$. (The Girardin/Elkies formula can be derived from taking $\alpha = i$ and $y \mapsto iy$.) Note that the factor of α is on the outside; one can put $\alpha^{1/3}$ inside, and also note that p_α is a polynomial in x^3 and y^3.

For simplicity, rewrite (1) as $F_{1,\alpha}^3 + F_{2,\alpha}^3 = p_\alpha$. Now let

$$
\begin{aligned}
F_{3,\alpha}(x, y) &= F_{1,\alpha}(\omega x, \omega^2 y), \quad & F_{4,\alpha}(x, y) &= F_{2,\alpha}(\omega x, \omega^2 y), \\
F_{5,\alpha}(x, y) &= F_{1,\alpha}(\omega^2 x, \omega y), \quad & F_{6,\alpha}(x, y) &= F_{2,\alpha}(\omega^2 x, \omega y),
\end{aligned}
$$

There is one master identity:

$$(\alpha x^2 - xy + \alpha y^2)^3 + \alpha(-x^2 + \alpha xy - y^2)^3 \\ = (\alpha^2 - 1)(\alpha x^3 + y^3)(x^3 + \alpha y^3) := p_\alpha(x, y). \tag{1}$$

This is "non-trivial" as long as $\alpha \notin \{0, \pm 1\}$. (The Girardin/Elkies formula can be derived from taking $\alpha = i$ and $y \mapsto iy$.) Note that the factor of α is on the outside; one can put $\alpha^{1/3}$ inside, and also note that p_α is a polynomial in x^3 and y^3.

For simplicity, rewrite (1) as $F_{1,\alpha}^3 + F_{2,\alpha}^3 = p_\alpha$. Now let

$$F_{3,\alpha}(x, y) = F_{1,\alpha}(\omega x, \omega^2 y), \quad F_{4,\alpha}(x, y) = F_{2,\alpha}(\omega x, \omega^2 y),$$
$$F_{5,\alpha}(x, y) = F_{1,\alpha}(\omega^2 x, \omega y), \quad F_{6,\alpha}(x, y) = F_{2,\alpha}(\omega^2 x, \omega y),$$

so that we have a "natural" three-fold identity:

$$F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3 = F_{5,\alpha}^3 + F_{6,\alpha}^3 = p_\alpha. \tag{2}$$

There is one master identity:

$$(\alpha x^2 - xy + \alpha y^2)^3 + \alpha(-x^2 + \alpha xy - y^2)^3 \\ = (\alpha^2 - 1)(\alpha x^3 + y^3)(x^3 + \alpha y^3) := p_\alpha(x, y). \quad (1)$$

This is "non-trivial" as long as $\alpha \notin \{0, \pm 1\}$. (The Girardin/Elkies formula can be derived from taking $\alpha = i$ and $y \mapsto iy$.) Note that the factor of α is on the outside; one can put $\alpha^{1/3}$ inside, and also note that p_α is a polynomial in x^3 and y^3.

For simplicity, rewrite (1) as $F_{1,\alpha}^3 + F_{2,\alpha}^3 = p_\alpha$. Now let

$$F_{3,\alpha}(x, y) = F_{1,\alpha}(\omega x, \omega^2 y), \quad F_{4,\alpha}(x, y) = F_{2,\alpha}(\omega x, \omega^2 y),$$
$$F_{5,\alpha}(x, y) = F_{1,\alpha}(\omega^2 x, \omega y), \quad F_{6,\alpha}(x, y) = F_{2,\alpha}(\omega^2 x, \omega y),$$

so that we have a "natural" three-fold identity:

$$F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3 = F_{5,\alpha}^3 + F_{6,\alpha}^3 = p_\alpha. \quad (2)$$

Elliptic curve addition on $(F_{1,\alpha}, F_{2,\alpha}, F_{3,\alpha}, F_{4,\alpha})$ yields two rational functions whose cubes sum to p_α. But it turns out that the denominators divide the numerators and we obtain $(F_{5,\alpha}, F_{6,\alpha})$ (!).

We can now present the main results. First, some notation. Suppose $q_1^3 + q_2^3 = q_3^3 + q_4^3$. There are 4! ways to permute the summands (taking sign into account) and then 3^4 ways to throw in powers of ω. We call any of these 1944 identities a *version* of the original equation. One such is $q_3^3 + (-\omega^2 q_2)^3 = (-\omega q_4)^3 + (\omega q_1)^3$.

We can now present the main results. First, some notation. Suppose $q_1^3 + q_2^3 = q_3^3 + q_4^3$. There are 4! ways to permute the summands (taking sign into account) and then 3^4 ways to throw in powers of ω. We call any of these 1944 identities a *version* of the original equation. One such is $q_3^3 + (-\omega^2 q_2)^3 = (-\omega q_4)^3 + (\omega q_1)^3$.

Theorem (Main Results)

(1) Suppose $q_j \in \mathbb{C}[x,y]$ are pairwise non-proportional quadratic forms and $q_1^3 + q_2^3 = q_3^3 + q_4^3$. Then there is an invertible linear change of variables which takes some version into $F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3$ for a computable α.

We can now present the main results. First, some notation. Suppose $q_1^3 + q_2^3 = q_3^3 + q_4^3$. There are 4! ways to permute the summands (taking sign into account) and then 3^4 ways to throw in powers of ω. We call any of these 1944 identities a *version* of the original equation. One such is $q_3^3 + (-\omega^2 q_2)^3 = (-\omega q_4)^3 + (\omega q_1)^3$.

Theorem (Main Results)

(1) Suppose $q_j \in \mathbb{C}[x, y]$ are pairwise non-proportional quadratic forms and $q_1^3 + q_2^3 = q_3^3 + q_4^3$. Then there is an invertible linear change of variables which takes some version into $F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3$ for a computable α.
(2) There is a version $g_1^3 + g_2^3 = g_3^3 + g_4^3$ so that there exists $T \in \mathbb{C}$ so that $g_1 + g_2 = T(g_3 + g_4)$; the other two flipped cases are the ones with the third representation. (For the $F_{k,\alpha}$'s, $T = \alpha^{2/3}$.)

We can now present the main results. First, some notation. Suppose $q_1^3 + q_2^3 = q_3^3 + q_4^3$. There are 4! ways to permute the summands (taking sign into account) and then 3^4 ways to throw in powers of ω. We call any of these 1944 identities a *version* of the original equation. One such is $q_3^3 + (-\omega^2 q_2)^3 = (-\omega q_4)^3 + (\omega q_1)^3$.

Theorem (Main Results)

(1) Suppose $q_j \in \mathbb{C}[x, y]$ are pairwise non-proportional quadratic forms and $q_1^3 + q_2^3 = q_3^3 + q_4^3$. Then there is an invertible linear change of variables which takes some version into $F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3$ for a computable α.
(2) There is a version $g_1^3 + g_2^3 = g_3^3 + g_4^3$ so that there exists $T \in \mathbb{C}$ so that $g_1 + g_2 = T(g_3 + g_4)$; the other two flipped cases are the ones with the third representation. (For the $F_{k,\alpha}$'s, $T = \alpha^{2/3}$.)
(3) There exists an invertible linear change taking the q_j's to $\mathbb{Q}[x, y]$ if and only if $T = a^2 + 3b^2$ for some $a, b \in \mathbb{Q}$.

Another general result is that after the linear change

Another general result is that after the linear change

$$(x, y) \mapsto \frac{1}{\sqrt{(1-\alpha^2)}}(\alpha x + y, -(x + \alpha y)),$$

Another general result is that after the linear change

$$(x,y) \mapsto \frac{1}{\sqrt{(1-\alpha^2)}}(\alpha x + y, -(x+\alpha y)),$$

$F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3 = F_{5,\alpha}^3 + F_{6,\alpha}^3 = p_\alpha$ is mapped to:

$$F_{7,\alpha}^3 + F_{8,\alpha}^3 = -F_{3,\alpha}^3 + F_{6,\alpha}^3 = -F_{5,\alpha}^3 + F_{4,\alpha}^3$$
$$= 3\sqrt{-3}\,\alpha xy(x^2-y^2)(\alpha x + y)(x+\alpha y).$$

Another general result is that after the linear change

$$(x, y) \mapsto \frac{1}{\sqrt{(1-\alpha^2)}}(\alpha x + y, -(x + \alpha y)),$$

$F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3 = F_{5,\alpha}^3 + F_{6,\alpha}^3 = p_\alpha$ is mapped to:

$$F_{7,\alpha}^3 + F_{8,\alpha}^3 = -F_{3,\alpha}^3 + F_{6,\alpha}^3 = -F_{5,\alpha}^3 + F_{4,\alpha}^3$$
$$= 3\sqrt{-3}\ \alpha xy(x^2 - y^2)(\alpha x + y)(x + \alpha y).$$

Here,

$$F_7(x, y) = \frac{1}{1-\alpha^2}\left((2\alpha + \alpha^3)x^2 + (1 + 5\alpha^2)xy + (2\alpha + \alpha^3)y^2\right);$$
$$F_8(x, y) = -\frac{1}{1-\alpha^2}\left((1 + 2\alpha^2)x^2 + (5\alpha + \alpha^3)xy + (1 + 2\alpha^2)y^2\right).$$

In order to make this seem more familiar, we return to (2) and make the linear change $(x, y) \mapsto (x + \omega^2 y, x + \omega y)$, and write $\alpha = \lambda^3$, absorbing the factor into the cube. We ultimately obtain, after some shuffling,

In order to make this seem more familiar, we return to (2) and make the linear change $(x, y) \mapsto (x + \omega^2 y, x + \omega y)$, and write $\alpha = \lambda^3$, absorbing the factor into the cube. We ultimately obtain, after some shuffling,

$$
\begin{aligned}
&(mx^2 - mxy - py^2)^3 + (nx^2 - nxy + \ell y^2)^3 \\
&= (-px^2 - mxy + my^2)^3 + (\ell x^2 - nxy + ny^2)^3 \\
&= (-px^2 + (m + 2p)xy - py^2)^3 + (\ell x^2 + (n - 2\ell)xy + \ell y^2)^3,
\end{aligned}
$$

In order to make this seem more familiar, we return to (2) and make the linear change $(x, y) \mapsto (x + \omega^2 y, x + \omega y)$, and write $\alpha = \lambda^3$, absorbing the factor into the cube. We ultimately obtain, after some shuffling,

$$(mx^2 - mxy - py^2)^3 + (nx^2 - nxy + \ell y^2)^3$$
$$= (-px^2 - mxy + my^2)^3 + (\ell x^2 - nxy + ny^2)^3$$
$$= (-px^2 + (m + 2p)xy - py^2)^3 + (\ell x^2 + (n - 2\ell)xy + \ell y^2)^3,$$

where

$$\ell = \lambda(\lambda^3 + 1), \quad m = 2\lambda^3 - 1, \quad n = \lambda(\lambda^3 - 2), \quad p = \lambda^3 + 1.$$

In order to make this seem more familiar, we return to (2) and make the linear change $(x, y) \mapsto (x + \omega^2 y, x + \omega y)$, and write $\alpha = \lambda^3$, absorbing the factor into the cube. We ultimately obtain, after some shuffling,

$$(mx^2 - mxy - py^2)^3 + (nx^2 - nxy + \ell y^2)^3$$
$$= (-px^2 - mxy + my^2)^3 + (\ell x^2 - nxy + ny^2)^3$$
$$= (-px^2 + (m+2p)xy - py^2)^3 + (\ell x^2 + (n-2\ell)xy + \ell y^2)^3,$$

where

$$\ell = \lambda(\lambda^3 + 1), \quad m = 2\lambda^3 - 1, \quad n = \lambda(\lambda^3 - 2), \quad p = \lambda^3 + 1.$$

And yes, this recovers the Narayanan equations with a bonus extra sum of two cubes.

It is hopeless to complete the proof of this in an hour, let alone twenty minutes, so let me sketch some of the elementary and 19th century style ingredients in the proof.

It is hopeless to complete the proof of this in an hour, let alone twenty minutes, so let me sketch some of the elementary and 19th century style ingredients in the proof.

Theorem (Maybe mine)

If $p \in \mathbb{C}[x_1, \ldots, x_n]$, then there exist $f, g \in \mathbb{C}[x_1, \ldots, x_n]$, such that $p = f^3 + g^3$ if and only if p is a cube, or if $p = q_1 q_2 q_3$, where q_i's are pairwise non-proportional, but linearly dependent.

It is hopeless to complete the proof of this in an hour, let alone twenty minutes, so let me sketch some of the elementary and 19th century style ingredients in the proof.

Theorem (Maybe mine)

If $p \in \mathbb{C}[x_1, \dots, x_n]$, then there exist $f, g \in \mathbb{C}[x_1, \dots, x_n]$, such that $p = f^3 + g^3$ if and only if p is a cube, or if $p = q_1 q_2 q_3$, where q_i's are pairwise non-proportional, but linearly dependent.

The proof depends on two easily verified facts:

It is hopeless to complete the proof of this in an hour, let alone twenty minutes, so let me sketch some of the elementary and 19th century style ingredients in the proof.

Theorem (Maybe mine)

If $p \in \mathbb{C}[x_1, \ldots, x_n]$, then there exist $f, g \in \mathbb{C}[x_1, \ldots, x_n]$, such that $p = f^3 + g^3$ if and only if p is a cube, or if $p = q_1 q_2 q_3$, where q_i's are pairwise non-proportional, but linearly dependent.

The proof depends on two easily verified facts:

$$f^3 + g^3 = (f + g)(f + \omega g)(f + \omega^2 g);$$

It is hopeless to complete the proof of this in an hour, let alone twenty minutes, so let me sketch some of the elementary and 19th century style ingredients in the proof.

Theorem (Maybe mine)

If $p \in \mathbb{C}[x_1, \dots, x_n]$, then there exist $f, g \in \mathbb{C}[x_1, \dots, x_n]$, such that $p = f^3 + g^3$ if and only if p is a cube, or if $p = q_1 q_2 q_3$, where q_i's are pairwise non-proportional, but linearly dependent.

The proof depends on two easily verified facts:

$$f^3 + g^3 = (f + g)(f + \omega g)(f + \omega^2 g);$$

$FG(aF + bG)$

It is hopeless to complete the proof of this in an hour, let alone twenty minutes, so let me sketch some of the elementary and 19th century style ingredients in the proof.

Theorem (Maybe mine)

If $p \in \mathbb{C}[x_1, \dots, x_n]$, then there exist $f, g \in \mathbb{C}[x_1, \dots, x_n]$, such that $p = f^3 + g^3$ if and only if p is a cube, or if $p = q_1 q_2 q_3$, where q_i's are pairwise non-proportional, but linearly dependent.

The proof depends on two easily verified facts:

$$f^3 + g^3 = (f + g)(f + \omega g)(f + \omega^2 g);$$

$$FG(aF + bG) = \frac{(aF - \omega^2 bG)^3 - (aF - \omega bG)^3}{3ab(\omega - \omega^2)}, \quad ab \neq 0.$$

The second result is old and I can't find a proof in a book younger than me.

Theorem

If $f(x, y)$ and $g(x, y)$ are two relative prime binary quadratic forms in $\mathbb{C}[x, y]$, then they may be simultaneously diagonalized; that is, there exist $a, b, c, d, ad \neq bc$, so that the coefficients of xy in each of $f(ax + by, cx + dy)$ and $g(ax + by, cx + dy)$ vanish.

The second result is old and I can't find a proof in a book younger than me.

Theorem

If $f(x,y)$ and $g(x,y)$ are two relative prime binary quadratic forms in $\mathbb{C}[x,y]$, then they may be simultaneously diagonalized; that is, there exist a,b,c,d, $ad \neq bc$, so that the coefficients of xy in each of $f(ax+by,cx+dy)$ and $g(ax+by,cx+dy)$ vanish.

The third result is undergraduate.

The second result is old and I can't find a proof in a book younger than me.

Theorem

If $f(x, y)$ and $g(x, y)$ are two relative prime binary quadratic forms in $\mathbb{C}[x, y]$, then they may be simultaneously diagonalized; that is, there exist $a, b, c, d, ad \neq bc$, so that the coefficients of xy in each of $f(ax + by, cx + dy)$ and $g(ax + by, cx + dy)$ vanish.

The third result is undergraduate.

Theorem

If $q_1^3 + q_2^3 = q_3^3 + q_4^3$ for four binary quadratic forms q_i , then there exist c_i, not all zero, so that $\sum c_i q_i = 0$.

The second result is old and I can't find a proof in a book younger than me.

Theorem

If $f(x, y)$ and $g(x, y)$ are two relative prime binary quadratic forms in $\mathbb{C}[x, y]$, then they may be simultaneously diagonalized; that is, there exist $a, b, c, d, ad \neq bc$, so that the coefficients of xy in each of $f(ax + by, cx + dy)$ and $g(ax + by, cx + dy)$ vanish.

The third result is undergraduate.

Theorem

If $q_1^3 + q_2^3 = q_3^3 + q_4^3$ for four binary quadratic forms q_i , then there exist c_i, not all zero, so that $\sum c_i q_i = 0$.

Note (crucially!) that the c_i's don't change when you make a change of variables.

Here is a broad sketch of the proof: If $p = q_1^3 + q_2^3 = q_3^3 + q_4^3$ is a non-trivial identity for binary quadratic forms q_i, first show that $\gcd(q_1, q_2) = 1$. Effect a linear change of variables that diagonalizes q_1 and q_2. Since q_1 and q_2 are even, so is p. After eliminating some cases, we can write

$$p(x,y) = (x^2 - r^2y^2)(x^2 - s^2y^2)(x^2 - t^2y^2).$$

Here is a broad sketch of the proof: If $p = q_1^3 + q_2^3 = q_3^3 + q_4^3$ is a non-trivial identity for binary quadratic forms q_i, first show that $\gcd(q_1, q_2) = 1$. Effect a linear change of variables that diagonalizes q_1 and q_2. Since q_1 and q_2 are even, so is p. After eliminating some cases, we can write

$$p(x, y) = (x^2 - r^2y^2)(x^2 - s^2y^2)(x^2 - t^2y^2).$$

There are 15 ways to write p as a product of three quadratics, and check when these are linearly dependent. This gives conditions on (r, s, t) and, with some truly awful computations involving linear changes of variable, these lead to the Main Theorem.

Here is a broad sketch of the proof: If $p = q_1^3 + q_2^3 = q_3^3 + q_4^3$ is a non-trivial identity for binary quadratic forms q_i, first show that $\gcd(q_1, q_2) = 1$. Effect a linear change of variables that diagonalizes q_1 and q_2. Since q_1 and q_2 are even, so is p. After eliminating some cases, we can write

$$p(x, y) = (x^2 - r^2y^2)(x^2 - s^2y^2)(x^2 - t^2y^2).$$

There are 15 ways to write p as a product of three quadratics, and check when these are linearly dependent. This gives conditions on (r, s, t) and, with some truly awful computations involving linear changes of variable, these lead to the Main Theorem.

Final remark: it's not so hard to solve the pair of equations $f_1^3 + f_2^3 = f_3^3 + f_4^3$ and $f_1 + f_2 = T(f_3 + f_4)$. You can start by dividing them to get $f_1^2 - f_1f_2 + f_2^2 = T^{-1}(f_3^2 - f_3f_4 + f_4^2)$, which leads to $f_1 + f_2$ and f_1f_2 as quadratics in f_3, f_4.

Thank you for your attention!

Thank you for your attention!

I'd like to thank the organizers for the invitation to speak and for their generous flexibility in letting me Zoom into this Session.

Thank you for your attention!

I'd like to thank the organizers for the invitation to speak and for their generous flexibility in letting me Zoom into this Session.

I'd also like to thank Bruce Berndt once again.