Discrete zeros of real ternary psd forms

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In this talk, we are concerned with $P_{3,2k}$, the cone of psd real $2k$-ic ternary forms: i.e., homogeneous polynomials $p(x, y, z)$ of degree $2k$ with the property that $p(a, b, c) \geq 0$ for $(a, b, c) \in \mathbb{R}^3$. We'll also be interested in the subcone $\Sigma_{3,2k}$ consisting of sums of squares of ternary forms of degree k.

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For such forms, we are particularly interested in the zero set of p, written $\mathcal{Z}(p)$, and the *projective* number of zeros, $|\mathcal{Z}(p)|$, counted this way because forms vanish on lines through the origin. We will describe $\mathcal{Z}(p)$ by picking a representative from each such line.

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If e.g. $p(x,y,z) = (x-z)^2 q(x,y,z)$ for some psd q, then the entire line $\{x = z\}$ is contained in $\mathcal{Z}(p)$, so $|\mathcal{Z}(p)| = \infty$. We will only be interested in those cases where $|\mathcal{Z}(p)|$ is finite, so we assume no indefinite square factors.

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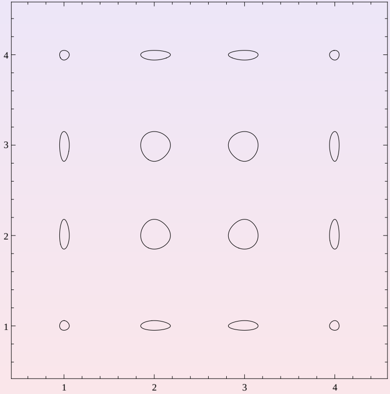
If $p(a,b,0) = 0$, then p has a "zero at infinity". In the absence of these, it makes sense to dehomogenize to $p(x,y,1)$. If $\epsilon > 0$ is sufficiently small, then the real solutions to $p(x,y) = \epsilon$ will consist of $|\mathcal{Z}(p)|$ disjoint ovals in the plane, one around each of the zeros.

Here is one example of 16 ovals for the octic q_4:

In[6]:=
```
ContourPlot[Product[(x - i)^2, {i, 1, 4}] +
   Product[(y - j)^2, {j, 1, 4}] == .1, {x, .5, 4.5},
  {y, .5, 4.5}, ContourStyle → Black, PlotPoints → 100]
```

Out[6]=



M. D. Choi, T. Y. Lam and I studied this topic systematically in 1980. Here are some of our results:

- There is an integer $\alpha(2k)$ with the property that if $p \in P_{3,2k}$ and $|\mathcal{Z}(p)| > \alpha(2k)$, then there exists an indefinite form h so that $p = h^2 q$. (If p is irreducible over $\mathbb{C}$ and $p(\pi) = 0$, then p is singular at π, and p has at most $(k-1)(2k-1)$ singular points; four variable fail: $x^2y^2 + z^2w^2$!)

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- $\alpha(2rk) \ge r^2\alpha(2k)$. (Argument to follow.)

Examples. If p is a real ternary form of degree $2k = 2, 4$, then psd implies sos, so the upper bounds are $1^2, 2^2$. These are achieved by:

$$\mathcal{Z}(x^2 + y^2 + z^2 - xy - xz - yz) = \{(1, 1, 1)\};$$
$$\mathcal{Z}(x^4 + y^4 + z^4 - x^2y^2 - x^2z^2 - y^2z^2) = \{(\pm 1, \pm 1, 1)\}.$$

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It turns out that $R := F^2 + G^2 + K$ is psd and has the original 8 zeros plus 2 at infinity. Miraculously, R is symmetric in x, y, z, even though z was treated differently from x and y.

Here are some dehomogenized ($z = 1$) pictures. This shows the set $F^2 + G^2 = .1$.

```
In[7]:= ContourPlot[x^2 (x^2 - 1)^2 + y^2 (y^2 - 1)^2 == .1,
  {x, -1.5, 1.5}, {y, -1.5, 1.5},
  ContourStyle → Black, PlotPoints → 100]
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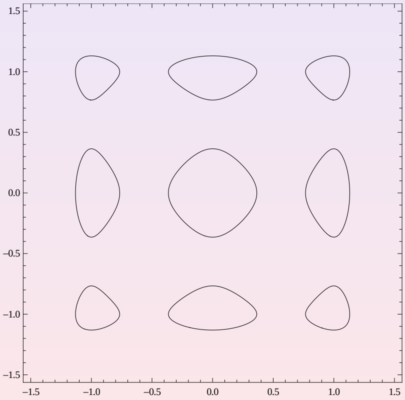
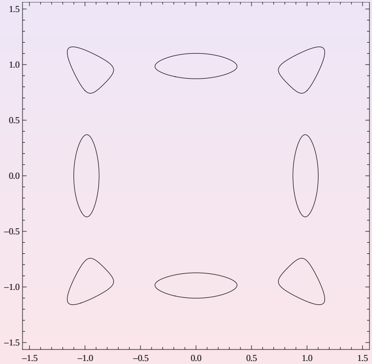
Out[7]=



This shows the set $R = F^2 + G^2 + K = .1$. You can't see the zeros at infinity.

```
In[8]:= ContourPlot[x^2 (x^2 - 1)^2 + y^2 (y^2 - 1)^2 +
     (x^2 - 1) (y^2 - 1) (1 - x^2 - y^2) == .1, {x, -1.5, 1.5},
   {y, -1.5, 1.5}, ContourStyle → Black, PlotPoints → 100]
```

Out[8]=



After algebraic simplification,

$$R(x,y,z) = x^6 + y^6 + z^6$$
$$-(x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4)$$
$$+3x^2y^2z^2.$$

We have

$$\mathcal{Z}(R) = \{(\pm 1, \pm 1, 1), (\pm 1, 0, 1), (0, \pm 1, 1), (1, \pm 1, 0)\}.$$

The last two zeros are at infinity; note that $|\mathcal{Z}(R)| = 10$ as promised. Both the singularity upper bound and the oval upper bound for sextics give 10, so $\alpha(6) = 10$.

Let $T_r(t) := \cos(r \arccos(t))$ be the r-th Chebyshev polynomial ($deg(T_r) = r$); e.g. $T_3(t) = 4t^3 - 3t$. Chebyshev polynomials have the property that $T_r : [-1,1] \mapsto [-1,1]$ in such a way that for $u \in (-1,1)$, the equation $T_r(t) = u$ has exactly r solutions.

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If $p \in P_{3,2k}$ and $|\mathcal{Z}(p)| = m$, then after an invertible linear change of variables, we may assume that

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We construct a new polynomial of degree $2kr$:

$$p_r(x,y,z) := z^{2kr} p(T_r(x/z), T_r(y/z), 1) \implies$$
$$\mathcal{Z}(p_r) = \{(T_r^{-1}(a_i), T_r^{-1}(b_i), 1) : 1 \le i \le m\},$$

so we see that $|\mathcal{Z}(p_r)| = r^2 m$. This gives quadratic growth.

Now some "new" results; this is joint work with Greg.

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The octic examples come from emulating Robinson's construction, but starting with a 4×4 grid. First ignore two zeros. It turns out that the set of quartics which vanish on these 14 points is a pencil with generators, say, F and G. We then look at octic forms which are singular at these 14 points. When we are lucky, they form a subspace of ternary octics with basis $\{F^2, FG, G^2, K\}$ for some K. We then play with taking $\phi(F, G) + \lambda K$ where ϕ is a pd quadratic form, and, when things work out just right, we find the examples.

The example with 17 zeros comes from a variation. We start with a 3×4 grid and a symmetric pair above and below.) The resulting $F_1(x, y, z)$ is unfortunately, quite ugly: $F_1 \in \mathbb{Q}(\sqrt{345})[x, y, z]$, and the three new zeros are at infinity; at $(0, 1, 0)$ and $(a, b, 0)$, where $3\sqrt{345}a^2 = 23b^2$. We have varied the starting points and found many similar examples, but none with rational coefficients.

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$$F_1(x,y,z) := -y^2(5x^2+9y^2-81z^2)(5x^2+y^2-9z^2)(y^2-4z^2)$$
$$+\frac{2}{27}(675+23\sqrt{345})x^2y^2(y^2-4z^2)^2$$
$$+9(5x^4-y^4-50x^2z^2+4y^2z^2+45z^4)^2$$

In 1893, Hilbert proved that if $p \in P_{3,2k}$ and $2k \geq 4$, then there exists $q \in P_{3,2k-4}$ so that $pq \in \Sigma_{3,4k-4}$ is a sum of *three* squares of forms of degree $2k - 2$.

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This example F_1 has the property that the only quadratic q (up to multiple) so that qF_1 is a sum of squares is

$$q_1(x,y,z) = 90x^2 + \sqrt{345}\ y^2 + 14\sqrt{345}\ z^2.$$

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Now we turn to the "morally 18 zero" example. It has 16 zeros, but two of them are "deep", with the polynomial vanishing to fourth order in a certain direction. In a geometric sense, this happens when two zeros coalesce at a point, and $16+2=18$.

The 14 zeros we start with are

$$\{(a, b, 1) : a, b \in \{\pm 1, \pm 3\}, (a, b) \neq (3, 3), (-3, -3)\};$$

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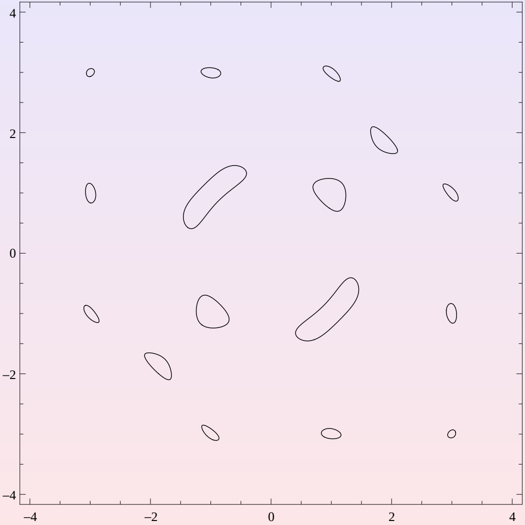
$$F_2(x,y,z) =$$
$$25x^8 + 72x^6y^2 + 144x^5y^3 + 194x^4y^4 + 144x^3y^5 + 72x^2y^6$$
$$+25y^8 - 572x^6z^2 - 144x^5yz^2 - 1436x^4y^2z^2 - 1728x^3y^3z^2$$
$$-1436x^2y^4z^2 - 144xy^5z^2 - 572y^6z^2 + 4192x^4z^4$$
$$+1584x^3yz^4 + 6584x^2y^2z^4 + 1584xy^3z^4$$
$$+4192y^4z^4 - 9720x^2z^6 - 1440xyz^6 - 9720y^2z^6 + 8100z^8$$

The next page shows $F_2(x,y,1) = 400$; 400 is small!

You can count 16 zeros and you can see the squeezed shape of the zeros at $(\pm 1, \mp 1)$, which is consistent with their 4th order.

Out[28]=



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$$W(x,y,z) = 16\sum x^{10} - 36\sum x^8y^2 + 20\sum x^6y^4$$
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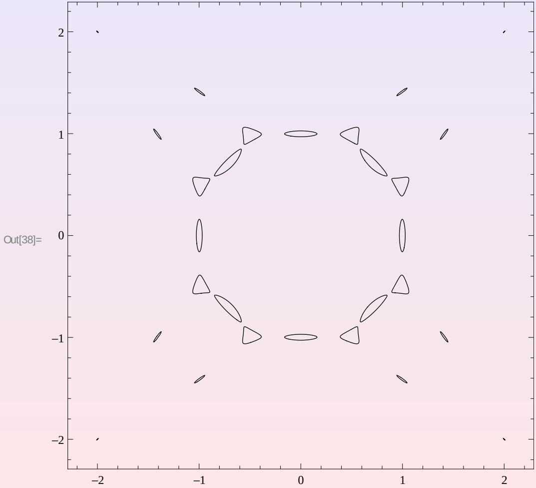
Harris showed that W is psd and $\mathcal{Z}(W)$ consists of $(1,1,\sqrt{2})$, $(1,1,\frac{1}{2})$, and $(1,1,0)$ with all choices of sign and permutation. This gives $12+12+6=30$ zeros, of which 28 zeros are not at infinity. (It seems likely that the future examples in higher degree will be symmetric.) The next page shows $W(x,y,1)=.08$.

The zeros are at $(\pm 1, \pm\frac{1}{2})$ $(\pm\frac{1}{2}, \pm 1)$, $(\pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{2}})$,$(\pm 1, \pm\sqrt{2})$, $(\pm\sqrt{2}, \pm 1)$,$(\pm 1, 0)$, $(0, \pm 1)$,$(\pm 2, \pm 2)$. The last 4 are barely visible, but choosing a larger ϵ makes the ovals coalesce.



On the conjecture, Choi, Lam and I remarked in 1980 that because of the Chebyshev-fueled quadratic growth,

$$\alpha(6s) \ge 10s^2,$$
$$\alpha(6s+2) \ge 10s^2 + 1,$$
$$\alpha(6s+4) \ge 10s^2 + 4.$$

This is already enough to prove that $\alpha(2k) \ge k^2 + 1$ for all but 18 cases: $6s + 2$ for $1 \le s \le 6$ and $6s + 4$ for $1 \le s \le 12$. The new information about $\alpha(8)$ and $\alpha(10)$ reduces the open cases to eight: $2k \in \{14, 22, 26, 28, 34, 38, 46, 58\}$.

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We think the conjecture is true. It's hard to believe that there's anything interesting about ternary forms of these degrees.

Finally, we mention one application, taken from my 1992 Memoir. Let $Q_{3,2k}$ be the closed cone of sums of $2k$-th powers of real linear forms; this is the dual cone to $P_{3,2k}$.

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If $\mathcal{Z}(p) = \{(a_i, b_i, c_i)\}$ and the $|\mathcal{Z}(p)|$ forms $\{(a_i x + b_i y + c_i z)^{2k}\}$ are linearly independent, then any expression of the form

$$\sum_{i=1}^{|\mathcal{Z}(p)|} \lambda_i (a_i x + b_i y + c_i z)^{2k}, \quad (\lambda_i > 0)$$

has no other expression as a sum of $2k$-th powers of linear forms.

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has no other expression as a sum of $2k$-th powers of linear forms. The *a priori* lower bound on "maximal width" is $\frac{(k+1)(k+2)}{2}$, which e.g. for $2k = 10$ is 21. It is easy to find sums of 10th powers of linear ternary forms which need 21 summands. The Harris example demonstrates the existence of forms needing 30 summands.

Thanks to the organizers for the invitation.

Thanks to the audience for your patience and your attention!