On elliptesque and hyperbolesque curves

Bruce Reznick
University of Illinois, Urbana-Champaign

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The solution set to $((y'')^{-2/3})''' = 0$ *consists of the non-degenerate conics.*

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First Proof.

We see that $(y'')^{-2/3} = c_0 + c_1 x + c_2 x^2$, hence, taking a_1, a_3, c_2, $\Delta = 4c_0c_2 - c_1^2$ all to be non-zero below,

$$y'' \in \left\{ a_0,\ \frac{1}{(a_0 + a_1 x)^{3/2}},\ \frac{1}{(a_2 + a_3 x)^3},\ \frac{1}{(c_0 + c_1 x + c_2 x^2)^{3/2}} \right\}.$$

Therefore, y equals $d_0 + d_1 x$ plus exactly one of

$$\left\{ b_0 x^2,\ (b_0 + b_1 x)^{1/2}, \frac{1}{b_0 + b_1 x},\ \frac{4}{\Delta}(c_0 + c_1 x + c_2 x^2)^{1/2} \right\}.$$

It is routine to check that the solutions to these equations yield precisely the non-degenerate conics. □

I will save you the trouble and tell you that

$$((y'')^{-2/3})''' = -\frac{2}{27}(y'')^{-11/3}(40(y''')^3 - 45y''y'''y'''' + 9(y'')^2y''''').$$

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Second Proof.

Suppose y is a conic. Then $(x^2, xy, y^2, x, y, 1)$ is perpendicular to (A, B, C, D, E, F), and hence so are $(2x, xy' + y, 2yy', 1, y', 0)$ and all successive derivatives. Take the first 6 such vectors and observe that they are dependent, so their determinant vanishes. The determinant is $4y''((40(y''')^3 - 45y''y'''y'''' + 9(y'')^2y''''')$. □

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It is with the second differential equation that Monge first proved this result in 1809. I'll tell you in the next section how I was led to the theorem. When I asked friends for help tracking these results down last summer, Tom Craven pointed to a 2006 article by Alain Lascoux, which had Monge's version and not Sylvester's, but cited his 1886 paper. Lascoux's and Sylvester's interests involve invariant theory and higher degree curves as well.

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Definition

A set $\mathcal{S}$ is *hyberbolesque* if the conic determined by any five points from $\mathcal{S}$ is a hyperbola.
A set $\mathcal{S}$ is *elliptesque* if the conic determined by any five points from $\mathcal{S}$ is an ellipse (or circle).

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Definition (Equivalent definitions)

A set $\mathcal{S}$ is *hyberbolesque* if no parabola, circle or ellipse intersects $\mathcal{S}$ in five or more points. A set $\mathcal{S}$ is *elliptesque* if no parabola or hyperbola intersects $\mathcal{S}$ in five or more points.

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Any subset of a suitable conic section is hyperbolesque or elliptesque.

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$\{y = x^{3/2},\ x \in [1, (\frac{403}{79})^{1/5} \approx 1.385)$ is elliptesque. The upper bound is not sharp. However, if you take four points very close to $x = 1$ and the fifth at $(x, x^{3/2}$, where $x > 34 + 24\sqrt{2} \approx 67.94$, then you get a hyperbola.

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There are also some conjectural hyperbolesque and elliptesque sets, at least based on numerical experiments, as we'll see later.

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The idea is to find a null vector (A, B, C, D, E, F) to the 5×6 matrix whose rows are $F(x_1, y_1), \dots, F(x_5, y_5)$ for any five points $(x_j, y_j) \in \mathcal{S}$. This gives the coefficients as polynomials in the variables x_j, y_j, and one wants to look at the sign of $4AC/B^2$. Mathematica helpfully tells us that (up to a square), this quantity is a form of degree 12 in the 10 variables, and has 14694 terms.

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As a geometric condition, the following seems to be true. Any four general points in the plane lie on two parabolas, which have an unambiguous "inside" and "outside". If the fifth point is chosen inside both or outside both parabolas, you get a hyperbola; inside one and outside the other yields an ellipse.

Another general geometric observation is that if $x \in \mathcal{S}$ is not on the boundary of the convex hull of $\mathcal{S}$, then by Caratheodory, it is in a triangle. Any set of 5 points containing x and the vertices of that triangle will determine a hyperbola.

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The most obvious first computational choice is to look at $y = x^m$, in which case the 5×5 minors of the linear system are members of the Vandermonde family. In fact, if $y = x^3$ or $y = x^4$, then the coefficients of A and C all have the same sign and it is immediate that $AC < 0$ if $x_i \geq 0$. In the case of $y = x^{3/2}$, parameterized by $(x, y) = (t^2, t^3)$, the coefficients are of mixed sign and a crude lower bound for $t = t_i \in [1, M]$ is $403 - 79M^{10}$, from which the bound on the last page is derived.

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Thus, if $\mathcal{S}$ is a connected curve, it is part of the boundary of a convex set.

There is also a local condition. Suppose $y = y(x)$ has a power series expansion at x_0. We can take five points in a neighborhood of x_0 and find the series expansion of $4AC - B^2$ in terms of the data for y at x_0.

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If $y = y(x)$ is smooth and elliptesque (resp. hyperbolesque) for $x \in (a, b)$, then $3y''y'''' - 5(y''')^2 \geq 0$ (resp. ≤ 0) on (a, b).

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At this point, you shouldn't be surprised that

$$((y'')^{-2/3})'' = -\frac{2}{9}(y'')^{-8/3}(3y''y'''' - 5(y''')^2).$$

Note the negative sign.

If $y = x^t$, then it is simple to calculate that $3y''y'''' - 5(y''')^2 > 0$ for $t \in (\frac{1}{2}, 1) \cup (1, 2)$ and < 0 for $t \in (0, \frac{1}{2}) \cup (2, \infty)$. But as noted earlier for $y = x^{3/2}$, this necessary local condition is not sufficient if the five points are not all taken infinitesimally close.

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It turns out that $((y'')^{-2/3})'' = 0$ is a differential equation solved by all parabolas (and only them). More computations led directly to the Sylvester theorem; in particular, because $((y'')^{-2/3})''$ seemed to be constant on conics:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies ((y'')^{-2/3})'' = -\frac{2}{(ab)^{2/3}};$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \implies ((y'')^{-2/3})'' = \frac{2}{(ab)^{2/3}}.$$

Theorem ("Theoremesque"; proved in last 24 hours)

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Proof.

Assume wlog that there is a horizontal tangent at the origin and that $y''(0) = 2a_2 > 0$. Since the curve bounds a convex region we take the other tangent at (r, s) with $s > 0$. Take the five points to be $(-\epsilon, y(-\epsilon))$, $(0, 0)$, $(\epsilon, y(\epsilon))$, $(r - \epsilon, s')$, $(r + \epsilon, s')$ where $s' > 0$. As $\epsilon \to 0$, these determine an ellipse. $\square$

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Theorem ("Theoremesque")

Suppose $\mathcal{S}$ is smooth and is tangent to the lines ℓ_1 at P_1 and ℓ_2 and P_2. Let Π be the parabola defined by the same tangencies. If $x \in \mathcal{S}$ and $\mathcal{S}$ is elliptesque (resp. hyperbolesque), then x lies inside (outside) Π.

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Theorem

If $y = x + a_2x^2 + a_3x^3 + \dots$ *for* $x \geq 0$ *and* $y = -x + b_2x^2 + b_3x^3 + \dots$ *for* $x \leq 0$ *and* $a_2, b_2 \geq 0$, *with* $a_2 + b_2 > 0$, *then the curve is neither elliptesque nor hyperbolesque.*

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Proof.

Taking $x = -2t, -t, 0, t, 2t$ gives a hyperbola; taking $x = -t, 0, t, 2t, 3t$ gives an ellipse when $a_2 > 0$, $x = -3t, -2t, -t, 0, t$ gives an ellipse when $b_2 > 0$. □

Numerical examples are somewhat suspicious, because strange things might happen on rather small sets. It's pretty easy to ask Mathematica to take ten million trials of 5 sets of random points $(f(t), g(t))$, but $10^{7/5} \approx 25$ isn't very large. With that caveat,

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Example

In 10 million trials of $\mathcal{S} = (t, e^t), t \in [-5, 5]$, $4AC/B^2$ is always negative.

A rectangle is neither hyperbolesque nor elliptesque. It is easy to get an ellipse which intersects it in 8 points and three points chosen from one edge and two from an adjacent edge give two intersecting lines determine a degenerate hyperbola, but that seems like cheating.

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It took me a while to find a genuine non-degenerate hyperbola which intersects a square in six points. In case you're interested, they are $-501 + 400x + 100x^2 + 400y - 400xy + 100y^2 = 0$ and the square with vertices $(\pm 1, \pm 1)$. The intersection points are $(\pm .1, 1), (1, \pm .1), (-1, .9), (.9, -1)$. Vince Matsko has some numerical work on related questions.

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I will conclude this talk with three conjectures, of ever-increasing uncertainty:

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Remark (Conjecture 2)

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Remark (Conjecture 2)

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Remark (Conjecture 3)

These ideas are worth pursuing.

Thanks to the organizers for the invitation and to the audience!