Equal sums of cubes of quadratic forms

Bruce Reznick
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Ramanujan Conference in Honor of Bruce Berndt
Urbana, Illinois       June 7, 2019
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You've heard of the X-Men. Bruce is one of the q-Men. (Like the X-Men, the q-Men are gender-diverse.)
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About twenty years ago, Bruce Berndt presented the following in our number theory seminar:

In 1913, Ramanujan posed to the Journal of the Indian Mathematical Society the following question: “Shew that

\[(6x^2 − 4xy + 4y^2)^3 = (3x^2 + 5xy − 5y^2)^3 + (4x^2 − 4xy + 6y^2)^3 + (5x^2 − 5xy − 3y^2)^3,\]

and find other quadratic expressions satisfying similar relations.”
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and find other quadratic expressions satisfying similar relations.”
Possibly by treating 3, 4, 5 and 6 as variables, in 1914, S. Narayanan gave the more general expression

\[(\ell x^2 - nxy + ny^2)^3 = (px^2 + mxy - my^2)^3 +
(nx^2 - nxy + \ell y^2)^3 + (mx^2 - mxy - py^2)^3,\]

where

\[\ell = \lambda(\lambda^3 + 1), \quad m = 2\lambda^3 - 1, \quad n = \lambda(\lambda^3 - 2), \quad p = \lambda^3 + 1.\]
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If you set \(\lambda = 2\) and divide by 3, you recover Ramanujan’s formula.
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\[\ell = \lambda(\lambda^3 + 1), \quad m = 2\lambda^3 - 1, \quad n = \lambda(\lambda^3 - 2), \quad p = \lambda^3 + 1.\]

If you set \(\lambda = 2\) and divide by 3, you recover Ramanujan’s formula. What seems like seven equations in \(\ell, m, n, p\) is actually only two: \(m^3 + n^3 = p^3 - \ell^3 = mp^2 + n\ell^2\), whose general solution, up to multiple and cube roots of unity, is parameterized above.
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First we have

\[(4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3\]
\[= (6x^2 - 4xy + 4y^2)^3 - (3x^2 + 5xy - 5y^2)^3\]
Ramanujan’s identity can be rewritten as two equal sums of two cubes in three different ways, and there are some pleasant surprises.

First we have

\[(4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3 = (6x^2 - 4xy + 4y^2)^3 - (3x^2 + 5xy - 5y^2)^3 = (6x^2 - 8xy + 6y^2)^3 - (3x^2 - 11xy + 3y^2)^3.\]
A different transposition also has a third representation:

\[(6x^2 - 4xy + 4y^2)^3 - (5x^2 - 5xy - 3y^2)^3\]

\[= (4x^2 - 4xy + 6y^2)^3 + (3x^2 + 5xy - 5y^2)^3\]
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Furthermore, this second set of identities can be derived from the first by making a unimodular linear change of variables:

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(x, y) \rightarrow \left( \frac{5x - 2y}{\sqrt{21}}, \frac{3x + 3y}{\sqrt{21}} \right).
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Alas, the third transposition does not have a third representation.
It turns out that these properties (of a third representation, and the equivalence under linear change), are not specific to Ramanujan’s example. One can also write down comparable versions for the Narayanan formulas, as we shall see.
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More to the point, up to changes of variable and taking $\lambda \in \mathbb{C}$, it turns out that the Narayanan formula *completely* describe the solution in binary quadratic forms to

$$q_1^3 + q_2^3 = q_3^3 + q_4^3.$$
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$$q_1^3 + q_2^3 = q_3^3 + q_4^3.$$

Our analysis comes from looking at the equation in quadratic forms over $\mathbb{C}$ rather than over $\mathbb{Q}$. 

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The simplest identity over $\mathbb{C}$ seems to have been largely overlooked. (With a linear change, it’s by Girardin (1910); in this form, the earliest I know is Elkies, 1995). It’s not so applicable over $\mathbb{Q}$.

$$(x^2 + xy - y^2)^3 + (x^2 - xy - y^2)^3 = 2(x^2)^3 - 2(y^2)^3.$$ 

Observe that there are two additional pairs of summands on the left hand side, if you send $y \mapsto \omega y$ and $y \mapsto \omega^2 y$, where $\omega$ is a primitive cube root of 1, so we really obtain four equal sums of pairs of cubes of binary quadratic forms. We don’t have time for it here, but, for sextics, the record number of different representations as a sum of two cubes is six, held by $xy(x^4 - y^4)$, up to changes of variable.
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We don’t have time for it here, but, for sextics, the record number of different representations as a sum of two cubes is six, held by $xy(x^4 - y^4)$, up to changes of variable.
There is one symmetric identity which contains everything we need.

\[
(\alpha x^2 - xy + \alpha y^2)^3 + \alpha(-x^2 + \alpha xy - y^2)^3
\]

\[
= (\alpha^2 - 1)(\alpha x^3 + y^3)(x^3 + \alpha y^3) := p_\alpha(x, y).
\]

This is “non-trivial” as long as \(\alpha \not\in \{0, \pm 1\}\). (The Girardin/Elkies formula can be derived from taking \(\alpha = i\) and \(y \mapsto iy\).) Note that the factor of \(\alpha\) is on the outside; one can put \(\alpha^{1/3}\) inside, and also note that \(p_\alpha\) is a polynomial in \(x^3\) and \(y^3\).
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\] (1)

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For simplicity, rewrite (1) as \( F_{3,\alpha}^1 + F_{3,\alpha}^2 = p_\alpha \). Now let

\[
F_{3,\alpha}(x, y) = F_{1,\alpha}(\omega x, \omega^2 y), \quad F_{4,\alpha}(x, y) = F_{2,\alpha}(\omega x, \omega^2 y), \\
F_{5,\alpha}(x, y) = F_{1,\alpha}(\omega^2 x, \omega y), \quad F_{6,\alpha}(x, y) = F_{2,\alpha}(\omega^2 x, \omega y),
\]

so that we have a “natural” three-fold identity:

\[
F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3 = F_{5,\alpha}^3 + F_{6,\alpha}^3 = p_\alpha.
\] (2)
Theorem (Main Result)

Suppose $q_j \in \mathbb{C}[x, y]$ are pairwise non-proportional quadratic forms and $q_1^3 + q_2^3 = q_3^3 + q_4^3$. Then there are invertible linear changes of variables that take two of the equations

\begin{align*}
q_1^3 + q_2^3 &= q_3^3 + q_4^3, \\
q_1^3 - q_3^3 &= q_4^3 - q_2^3, \\
q_1^3 - q_4^3 &= q_3^3 - q_2^3,
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into $F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3$ for a computable $\alpha$. 
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If \( q_j \in \mathbb{Q}[x, y] \), then \( \alpha^2 \) is a rational cube.
Another general result is that after the linear change

\[(x, y) \mapsto \frac{1}{\sqrt{1 - \alpha^2}}(\alpha x + y, -(x + \alpha y)),\]

\[F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3 = F_{5,\alpha}^3 + F_{6,\alpha}^3 = p_\alpha \text{ is mapped to:}\]

\[F_{7,\alpha}^3 + F_{8,\alpha}^3 = -F_{3,\alpha}^3 + F_{6,\alpha}^3 = -F_{5,\alpha}^3 + F_{4,\alpha}^3\]

\[= 3\sqrt{-3} \alpha xy(x^2 - y^2)(\alpha x + y)(x + \alpha y).\]

Here,

\[F_7(x, y) = \frac{1}{1 - \alpha^2} \left( (2\alpha + \alpha^3)x^2 + (1 + 5\alpha^2)xy + (2\alpha + \alpha^3)y^2 \right);\]

\[F_8(x, y) = -\frac{1}{1 - \alpha^2} \left( (1 + 2\alpha^2)x^2 + (5\alpha + \alpha^3)xy + (1 + 2\alpha^2)y^2 \right).\]
In order to make this seem more familiar, we return to (2) and make the linear change \((x, y) \mapsto (x + \omega^2 y, x + \omega y)\), and write \(\alpha = \lambda^3\), absorbing the factor into the cube. We ultimately obtain, after some shuffling,

\[
(mx^2 - mxy - py^2)^3 + (nx^2 - nxy + \ell y^2)^3 = (-px^2 - mxy + my^2)^3 + (\ell x^2 - nxy + ny^2)^3 = (-px^2 + (m + 2p)xy - py^2)^3 + (\ell x^2 + (n - 2\ell)xy + \ell y^2)^3,
\]

where

\[
\ell = \lambda(\lambda^3 + 1), \quad m = 2\lambda^3 - 1, \quad n = \lambda(\lambda^3 - 2), \quad p = \lambda^3 + 1.
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where

\[\ell = \lambda(\lambda^3 + 1), \quad m = 2\lambda^3 - 1, \quad n = \lambda(\lambda^3 - 2), \quad p = \lambda^3 + 1.\]

And yes, this recovers the Narayanan equations with a bonus extra sum of two cubes.
It is hopeless to complete the proof of this in an hour, let alone fifteen minutes, so let me sketch some of the elementary and 19th century style ingredients in the proof.

**Theorem**

If \( p \in \mathbb{C}[x_1, \ldots, x_n] \), then there exist \( f, g \in \mathbb{C}[x_1, \ldots, x_n] \), such that 
\[ p = f^3 + g^3 \] 
if and only if \( p \) is a cube, or if \( p = q_1 q_2 q_3 \), where \( q_i \)'s are pairwise non-proportional, but linearly dependent.
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If $p \in \mathbb{C}[x_1, \ldots, x_n]$, then there exist $f, g \in \mathbb{C}[x_1, \ldots, x_n]$, such that $p = f^3 + g^3$ if and only if $p$ is a cube, or if $p = q_1 q_2 q_3$, where $q_i$'s are pairwise non-proportional, but linearly dependent.

The proof depends on two easily verified facts:

\[
f^3 + g^3 = (f + g)(f + \omega g)(f + \omega^2 g);
\]

\[
FG(aF + bG) = \frac{(aF - \omega^2 bG)^3 - (aF - \omega bG)^3}{3ab(\omega - \omega^2)}, \quad ab \neq 0.
\]
The second result is old and I can’t find a proof younger than me.

**Theorem**

If \( f(x, y) \) and \( g(x, y) \) are two relative prime binary quadratic forms in \( \mathbb{C}[x, y] \), then they may be simultaneously diagonalized; that is, there exist \( a, b, c, d \), \( ad \neq bc \), so that the coefficients of \( xy \) in each of \( f(ax + by, cx + dy) \) and \( g(ax + by, cx + dy) \) vanish.

The third result is undergraduate.

**Theorem**

If \( q_3^1 + q_3^2 = q_3^3 + q_3^4 \) for four quadratic forms, then there exist \( c_i \), not all zero, so that \( \sum c_i q_i = 0 \).

Note that the \( c_i \)'s don’t change when you make a change of variables.
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**Theorem**

If \( q_1^3 + q_2^3 = q_3^3 + q_4^3 \) for four quadratic forms, then there exist \( c_i \), not all zero, so that \( \sum c_i q_i = 0 \).

Note that the \( c_i \)'s don’t change when you make a change of variables.
Here is a broad sketch of the proof: If \( p = q_1^3 + q_2^3 = q_3^3 + q_4^3 \) is a non-trivial identity for binary quadratic forms \( q_i \), first show that \( \gcd(q_1, q_2) = 1 \). Effect a linear change of variables that diagonalizes \( q_1 \) and \( q_2 \). This implies that \( p \) is now even. After eliminating some cases, we can write

\[
p(x, y) = (x^2 - r^2 y^2)(x^2 - s^2 y^2)(x^2 - t^2 y^2).
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There are 15 ways to write \( p \) as a product of three quadratics, and check when these are linearly dependent. This gives conditions on \((r, s, t)\) and, with some truly awful computations involving linear changes of variable, these prove the Main Theorem.
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Finally, the coefficients of the linear combination in \( F_{j, \alpha} \), \( 1 \leq j \leq 4 \) are in \( \{ \pm 1, \pm \alpha^{2/3} \} \), so if there is a linear change with a solution in \( \mathbb{Q}[x, y] \), \( \alpha^2 \) must be a rational cube.
There is of course a massive history of work on the equation $a^3 + b^3 = c^3 + d^3$ over $\mathbb{Q}$, and I can’t do it justice here. I haven’t found this three-fold nature of the solution. We finish with two of the standard approaches to the equation, and how these tie in.

The first is “point-addition” on an elliptic curve:

$$f^3_1 + f^3_2 = g^3_1 + g^3_2 = p \Rightarrow h^3_1 + h^3_2 = p,$$

where

$$h^3_1 = f^3_1 + t(g^3_1 - f^3_1),$$
$$h^3_2 = f^3_2 + t(g^3_2 - f^3_2),$$
$$t = f^2_1 g^1_2 + f^2_2 g^1_1 - f^3_1 - f^3_2 f^2_1 g^1_2 - f^1_2 g^2_1 + f^2_2 g^2_1 - f^2_2 g^2_2.$$

If we take $(f_1, f_2, g_1, g_2) = (F_1, \alpha, F_2, \alpha, F_3, \alpha, F_4, \alpha)$, then we know that $(h_1, h_2)$ will be rational functions whose cubes sum to $p \alpha$. In fact, the denominators cancel, and we obtain $(F_5, \alpha, F_6, \alpha)$ (!).
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\end{align*}
\]

where

\[
\begin{align*}
h_1 &= f_1 + t(g_1 - f_1), \quad h_2 = f_2 + t(g_2 - f_2), \\
t &= \frac{f_1^2 g_1 + f_2^2 g_2 - f_1^3 - f_2^3}{f_1^2 g_1 - f_1 g_1^2 + f_2^2 g_2 - f_2 g_2^2}.
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The first is “point-addition” on an elliptic curve:

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f_1^3 + f_2^3 = g_1^3 + g_2^3 = p \implies h_1^3 + h_2^3 = p, \quad \text{where}
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h_1 = f_1 + t(g_1 - f_1), \quad h_2 = f_2 + t(g_2 - f_2),
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t = \frac{f_1^2 g_1 + f_2^2 g_2 - f_1^3 - f_2^3}{f_1^2 g_1 - f_1 g_1^2 + f_2^2 g_2 - f_2 g_2^2}.
\]

If we take \((f_1, f_2, g_1, g_2) = (F_1, \alpha, F_2, \alpha, F_3, \alpha, F_4, \alpha)\), then we know that \((h_1, h_2)\) will be rational functions whose cubes sum to \(p_\alpha\). In fact, the denominators cancel, and we obtain \((F_5, \alpha, F_6, \alpha)\) (!).
This second result is in Hardy & Wright.

**Theorem (Euler-Binet)**

If $a^3 + b^3 = c^3 + d^3$ for $a, b, c, d \in \mathbb{Q}$, then there exist $r, s, t \in \mathbb{Q}$ so that

- $a = (1 - (r - 3s)(r^2 + 3s^2))t,$
- $b = ((r + 3s)(r^2 + 3s^2) - 1)t,$
- $c = (r + 3s - (r^2 + 3s^2)^2)t,$
- $d = ((r^2 + 3s^2)^2 - (r - 3s))t.$
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\begin{align*}
    a &= (1 - (r - 3s)(r^2 + 3s^2))t, \\
    b &= ((r + 3s)(r^2 + 3s^2) - 1)t, \\
    c &= (r + 3s - (r^2 + 3s^2)^2)t, \\
    d &= ((r^2 + 3s^2)^2 - (r - 3s))t.
\end{align*}
\]

So, $1^3 + 12^3 = 10^3 + 9^3$ comes from $(r, s, t) = \left(\frac{107}{266}, -\frac{111}{266}, -\frac{2527}{333}\right)$,
This second result is in Hardy & Wright.

**Theorem (Euler-Binet)**

If \( a^3 + b^3 = c^3 + d^3 \) for \( a, b, c, d \in \mathbb{Q} \), then there exist \( r, s, t \in \mathbb{Q} \) so that

\[
\begin{align*}
    a &= (1 - (r - 3s)(r^2 + 3s^2))t, \\
    b &= ((r + 3s)(r^2 + 3s^2) - 1)t, \\
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\end{align*}
\]

So, \( 1^3 + 12^3 = 10^3 + 9^3 \) comes from \((r, s, t) = \left( \frac{107}{266}, -\frac{111}{266}, -\frac{2527}{333} \right)\), but \(10^3 + (-1)^3 = (-9)^3 + 12^3\) comes from \((r, s, t) = \left( -\frac{3}{2}, \frac{1}{2}, 1 \right)\).
Only minor modifications are needed to show that the Euler-Binet parameterization also applies when $\mathbb{Q}$ is replaced by $\mathbb{C}(x_1, \ldots, x_n)$; the reliance on $\sqrt{-3} \notin \mathbb{Q}$ can be dealt with formally. Degree considerations show that one can’t hope to have $r, s, t$ be polynomials in order to obtain our solutions in (2); even after rearranging, the smallest denominator of $t$ contains $xy$. 

Finally, we combine Euler-Binet with point-addition:

$$a^3 + (−d)^3 = (−b)^3 + c^3 = e^3 + f^3,$$

where $e = (2r (r^2 + 3s^2) + 1)t$, $f = −(2r + (r^2 + 3s^2)^2)t$ and where $(e, f)$ is the “sum” of $(a, −d)$ and $(-b, c)$. 

An appeal to the three linear dependent factors theorem shows that this is the only bonus polynomial representation from Euler-Binet.
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\[\text{Bruce Reznick University of Illinois at Urbana-Champaign} \quad \text{Equal sums of cubes of quadratic forms}\]
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Thank you for your attention!
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But an extra thanks to Bruce Berndt!