Equal sums of cubes of quadratic forms

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Ramanujan Conference in Honor of Bruce Berndt
Urbana, Illinois June 7, 2019

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(Like the X-Men, the q-Men are gender-diverse.)

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In 1913, Ramanujan posed to the *Journal of the Indian Mathematical Society* the following question: "Shew that

$$(6x^2 - 4xy + 4y^2)^3 = (3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3,$$

and find other quadratic expressions satisfying similar relations."

Possibly by treating 3, 4, 5 and 6 as variables, in 1914, S. Narayanan gave the more general expression

$$(\ell x^2 - nxy + ny^2)^3 = (px^2 + mxy - my^2)^3 + (nx^2 - nxy + \ell y^2)^3 + (mx^2 - mxy - py^2)^3,$$

where

$$\ell = \lambda(\lambda^3 + 1), \quad m = 2\lambda^3 - 1, \quad n = \lambda(\lambda^3 - 2), \quad p = \lambda^3 + 1.$$

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What seems like seven equations in ℓ, m, n, p is actually only two: $m^3 + n^3 = p^3 - \ell^3 = mp^2 + n\ell^2$, whose general solution, up to multiple and cube roots of unity, is parameterized above.

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First we have

$$(4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3$$
$$= (6x^2 - 4xy + 4y^2)^3 - (3x^2 + 5xy - 5y^2)^3$$

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A different transposition also has a third representation:

$$(6x^2 - 4xy + 4y^2)^3 - (5x^2 - 5xy - 3y^2)^3$$
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Furthermore, this second set of identities can be derived from the first by making a unimodular linear change of variables:

$$(x, y) \to \left(\frac{5x - 2y}{\sqrt{21}}, \frac{3x + 3y}{\sqrt{21}}\right).$$

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Alas, the third transposition does not have a third representation.

It turns out that these properties (of a third representation, and the equivalence under linear change), are not specific to Ramanujan's example. One can also write down comparable versions for the Narayanan formulas, as we shall see.

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More to the point, up to changes of variable and taking $\lambda \in \mathbb{C}$, it turns out that the Narayanan formula *completely* describe the solution in binary quadratic forms to

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Our analysis comes from looking at the equation in quadratic forms over $\mathbb{C}$ rather than over $\mathbb{Q}$.

The simplest identity over $\mathbb{C}$ seems to have been largely overlooked. (With a linear change, it's by Girardin (1910); in this form, the earliest I know is Elkies, 1995). It's not so applicable over $\mathbb{Q}$.

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Observe that there are two additional pairs of summands on the left hand side, if you send $y \mapsto \omega y$ and $y \mapsto \omega^2 y$, where ω is a primitive cube root of 1, so we really obtain *four* equal sums of pairs of cubes of binary quadratic forms.

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We don't have time for it here, but, for sextics, the record number of different representations as a sum of two cubes is *six*, held by $xy(x^4 - y^4)$, up to changes of variable.

There is one symmetric identity which contains everything we need.

$$\begin{aligned}&(\alpha x^2 - xy + \alpha y^2)^3 + \alpha(-x^2 + \alpha xy - y^2)^3 \\ &= (\alpha^2 - 1)(\alpha x^3 + y^3)(x^3 + \alpha y^3) := p_\alpha(x, y).\end{aligned} \tag{1}$$

This is "non-trivial" as long as $\alpha \notin \{0, \pm 1\}$. (The Girardin/Elkies formula can be derived from taking $\alpha = i$ and $y \mapsto iy$.) Note that the factor of α is on the outside; one can put $\alpha^{1/3}$ inside, and also note that p_α is a polynomial in x^3 and y^3.

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For simplicity, rewrite (1) as $F_{1,\alpha}^3 + F_{2,\alpha}^3 = p_\alpha$. Now let

$$
\begin{aligned}
F_{3,\alpha}(x,y) &= F_{1,\alpha}(\omega x, \omega^2 y), \quad F_{4,\alpha}(x,y) = F_{2,\alpha}(\omega x, \omega^2 y), \\
F_{5,\alpha}(x,y) &= F_{1,\alpha}(\omega^2 x, \omega y), \quad F_{6,\alpha}(x,y) = F_{2,\alpha}(\omega^2 x, \omega y),
\end{aligned}
$$

so that we have a "natural" three-fold identity:

$$
F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3 = F_{5,\alpha}^3 + F_{6,\alpha}^3 = p_\alpha. \tag{2}
$$

Theorem (Main Result)

Suppose $q_j \in \mathbb{C}[x,y]$ are pairwise non-proportional quadratic forms and $q_1^3 + q_2^3 = q_3^3 + q_4^3$. Then there are invertible linear changes of variables that take two of the equations

$$q_1^3 + q_2^3 = q_3^3 + q_4^3,$$
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into $F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3$ for a computable α.

If $q_j \in \mathbb{Q}[x,y]$, then α^2 is a rational cube.

Another general result is that after the linear change

$$(x, y) \mapsto \frac{1}{\sqrt{(1-\alpha^2)}}(\alpha x + y, -(x + \alpha y)),$$

$F_{1,\alpha}^3 + F_{2,\alpha}^3 = F_{3,\alpha}^3 + F_{4,\alpha}^3 = F_{5,\alpha}^3 + F_{6,\alpha}^3 = p_\alpha$ is mapped to:

$$F_{7,\alpha}^3 + F_{8,\alpha}^3 = -F_{3,\alpha}^3 + F_{6,\alpha}^3 = -F_{5,\alpha}^3 + F_{4,\alpha}^3$$
$$= 3\sqrt{-3}\ \alpha xy(x^2 - y^2)(\alpha x + y)(x + \alpha y).$$

Here,

$$F_7(x, y) = \frac{1}{1-\alpha^2}\left((2\alpha + \alpha^3)x^2 + (1 + 5\alpha^2)xy + (2\alpha + \alpha^3)y^2\right);$$
$$F_8(x, y) = -\frac{1}{1-\alpha^2}\left((1 + 2\alpha^2)x^2 + (5\alpha + \alpha^3)xy + (1 + 2\alpha^2)y^2\right).$$

In order to make this seem more familiar, we return to (2) and make the linear change $(x, y) \mapsto (x + \omega^2 y, x + \omega y)$, and write $\alpha = \lambda^3$, absorbing the factor into the cube. We ultimately obtain, after some shuffling,

$$
\begin{aligned}
&(mx^2 - mxy - py^2)^3 + (nx^2 - nxy + \ell y^2)^3 \\
&= (-px^2 - mxy + my^2)^3 + (\ell x^2 - nxy + ny^2)^3 \\
&= (-px^2 + (m + 2p)xy - py^2)^3 + (\ell x^2 + (n - 2\ell)xy + \ell y^2)^3,
\end{aligned}
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where

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where

$$\ell = \lambda(\lambda^3 + 1), \quad m = 2\lambda^3 - 1, \quad n = \lambda(\lambda^3 - 2), \quad p = \lambda^3 + 1.$$

And yes, this recovers the Narayanan equations with a bonus extra sum of two cubes.

It is hopeless to complete the proof of this in an hour, let alone fifteen minutes, so let me sketch some of the elementary and 19th century style ingredients in the proof.

Theorem

If $p \in \mathbb{C}[x_1, \dots, x_n]$, then there exist $f, g \in \mathbb{C}[x_1, \dots, x_n]$, such that $p = f^3 + g^3$ if and only if p is a cube, or if $p = q_1 q_2 q_3$, where q_i's are pairwise non-proportional, but linearly dependent.

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The proof depends on two easily verified facts:

$$f^3 + g^3 = (f+g)(f+\omega g)(f+\omega^2 g);$$

$$FG(aF + bG) = \frac{(aF - \omega^2 bG)^3 - (aF - \omega bG)^3}{3ab(\omega - \omega^2)}, \quad ab \neq 0.$$

The second result is old and I can't find a proof younger than me.

Theorem

If $f(x, y)$ and $g(x, y)$ are two relative prime binary quadratic forms in $\mathbb{C}[x, y]$, then they may be simultaneously diagonalized; that is, there exist $a, b, c, d, ad \neq bc$, so that the coefficients of xy in each of $f(ax + by, cx + dy)$ and $g(ax + by, cx + dy)$ vanish.

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The third result is undergraduate.

Theorem

If $q_1^3 + q_2^3 = q_3^3 + q_4^3$ for four quadratic forms, then there exist c_i, not all zero, so that $\sum c_i q_i = 0$.

Note that the c_i's don't change when you make a change of variables.

Here is a broad sketch of the proof: If $p = q_1^3 + q_2^3 = q_3^3 + q_4^3$ is a non-trivial identity for binary quadratic forms q_i, first show that $\gcd(q_1, q_2) = 1$. Effect a linear change of variables that diagonalizes q_1 and q_2. This implies that p is now even. After eliminating some cases, we can write

$$p(x, y) = (x^2 - r^2 y^2)(x^2 - s^2 y^2)(x^2 - t^2 y^2).$$

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Finally, the coefficients of the linear combination in $F_{j,\alpha}, 1 \le j \le 4$ are in $\{\pm 1, \pm\alpha^{2/3}\}$, so if there is a linear change with a solution in $\mathbb{Q}[x, y]$, α^2 must be a rational cube.

There is of course a massive history of work on the equation $a^3 + b^3 = c^3 + d^3$ over $\mathbb{Q}$, and I can't do it justice here. I haven't found this three-fold nature of the solution. We finish with two of the standard approaches to the equation, and how these tie in.

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The first is "point-addition" on an elliptic curve:

$$f_1^3 + f_2^3 = g_1^3 + g_2^3 = p \implies h_1^3 + h_2^3 = p, \quad \text{where}$$
$$h_1 = f_1 + t(g_1 - f_1), \; h_2 = f_2 + t(g_2 - f_2),$$
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If we take $(f_1, f_2, g_1, g_2) = (F_{1,\alpha}, F_{2,\alpha}, F_{3,\alpha}, F_{4,\alpha})$, then we know that (h_1, h_2) will be rational functions whose cubes sum to p_α. In fact, the denominators cancel, and we obtain $(F_{5,\alpha}, F_{6,\alpha})$ (!).

This second result is in Hardy & Wright.

Theorem (Euler-Binet)

If $a^3 + b^3 = c^3 + d^3$ *for* $a, b, c, d \in \mathbb{Q}$*, then there exist* $r, s, t \in \mathbb{Q}$ *so that*

$$\begin{aligned} a &= (1 - (r - 3s)(r^2 + 3s^2))t, \\ b &= ((r + 3s)(r^2 + 3s^2) - 1)t, \\ c &= (r + 3s - (r^2 + 3s^2)^2)t, \\ d &= ((r^2 + 3s^2)^2 - (r - 3s))t. \end{aligned}$$

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So, $1^3 + 12^3 = 10^3 + 9^3$ comes from $(r, s, t) = \left(\frac{107}{266}, -\frac{111}{266}, -\frac{2527}{333}\right)$,

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So, $1^3 + 12^3 = 10^3 + 9^3$ comes from $(r, s, t) = \left(\frac{107}{266}, -\frac{111}{266}, -\frac{2527}{333}\right)$,
but $10^3 + (-1)^3 = (-9)^3 + 12^3$ comes from $(r, s, t) = \left(-\frac{3}{2}, \frac{1}{2}, 1\right)$.

Only minor modifications are needed to show that the Euler-Binet parameterization also applies when $\mathbb{Q}$ is replaced by $\mathbb{C}(x_1, \ldots, x_n)$; the reliance on $\sqrt{-3} \notin \mathbb{Q}$ can be dealt with formally. Degree considerations show that one can't hope to have r, s, t be polynomials in order to obtain our solutions in (2); even after rearranging, the smallest denominator of t contains xy .

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Finally, we combine Euler-Binet with point-addition:

$$a^3 + (-d)^3 = (-b)^3 + c^3 = e^3 + f^3,$$

where

$$e = (2r(r^2 + 3s^2) + 1)t, \quad f = -(2r + (r^2 + 3s^2)^2)t$$

and where (e, f) is the "sum" of $(a, -d)$ and $(-b, c)$.

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An appeal to the three linear dependent factors theorem shows that this is the only bonus polynomial representation from Euler-Binet.

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