MATH 595, SPRING 2012, SECOND EXERCISE SET

Due Monday, March 5, 2012. Again, make of these what you want. If you find computer programs (in any language) useful, please attach a copy of the code with your solutions.

1. (Cf. §2.1): If $ord(h) \geq 1$, show from the definition that

$$\sum_{n=0}^{\infty} h^n = (1-h)^{-1}.$$

2. (Cf. §2.6): Show that $S(3 \cdot 2^{r-1}) = 3^r + 1$ for $r \geq 1$, compute $S(5 \cdot 2^{r-2})$ and $S(7 \cdot 2^{r-2})$ and examine the implications for $n^{-\gamma} S(n)$ at $n = 5 \cdot 2^{r-2}, 7 \cdot 2^{r-2}$

3. (Cf. §2.7): Show the generating function for $\mathcal{U}$ over $\mathbb{Z}/2\mathbb{Z}$ is $\frac{1-X^3}{1-X} = 1 + X + X^2$.

4. Define the sequence $(a(n))$ by:

$$a(0) = 0,\ a(1) = 1; \qquad a(2n) = -a(n),\ a(2n+1) = a(n) + a(n+1) \quad \text{for } n \geq 1.$$

a. Calculate $a(n)$ for $n \leq 10$, derive a hypothesis for a general formula and prove it, eg, by induction.
b. Using your answer in a., find a simple expression for the generating function $\sum_{n=0}^{\infty} a(n) x^n$ as a rational function.
c. Use the method from the notes for the Stern sequence generating function to show that

$$\sum_{n=0}^{\infty} a(n) x^n = x \prod_{j=0}^{\infty} (1 - x^{2^j} + x^{2^{j+1}}).$$

Reconcile this infinite product with your answer in b.

5. Let $d(n)$ denote the number of ways to write $n = \sum_i \epsilon_i 2^i$, where $\epsilon_i \in \{0, 1, 3\}$. Determine a recurrence for $d(n)$ analogous to (1.85) and prove that $d(n+7) \equiv d(n) \pmod 2$. (This problem is taken from Melissa Dennison's 2010 thesis, and is greatly generalized in the Anders-Dennison-Lansing-Reznick paper in the Vault; Katie and Jennifer are excused from doing this problem!)

6. (Not a Stern question, but a recurrence question.) Let F_n denote the Fibonacci sequence and suppose a and b are distinct fixed positive integers. Determine constants $r = r(a,b), s = s(a,b)$ so that

$$F_{n+b} = rF_{n+a} + sF_n.$$

What can you say algebraically about the polynomial $x^{a+b} - rx^a - s$?

7. (Not a Stern question, but a recurrence question.) Suppose $x^2 + ax + b = 0$ has distinct roots and

$$y(n) + ay(n-1) + by(n-2) = 0, \quad \text{for } n \geq 2.$$

Find a third order liner recurrence satisfied by $(z(n)) = ((y(n))^2)$. Hint: this can be construed as an application of Newton's Theorem on symmetric polynomials.

For problems 8 and 9, let

$$B_k(r) = \sum_{n \in I_r}^{*} s(n)^k.$$

8. Using what you know about the formula for $B_k(r)$ for $k = 0, 1, 2$, find a third order recurrence for the sequence (c_r), given in HW1 #9. It doesn't really make sense to try to prove the recurrence directly.

9. Using the methods of class, find recurrences for $B_4(r)$ and $B_5(r)$. You do not have to solve the recurrences, but they aren't bad: the first one is third-order, but the characteristic polynomial has a simple linear factor; the second one is second-order for r sufficiently large.

10. (Extra credit) Find something new and interesting to say about the sequence $(u(n))$ from section 2.7.

11. (Extra credit) This is the rejected *Monthly* problem: Let F_n denote the n-th Fibonacci number. Prove that, for $n \geq 1$,

$$\sum_{k=1}^{n} (2F_k F_{k-1} + (-1)^{k-1})^3 = 2F_n^6 - \tfrac{1}{2}(1 + (-1)^{n-1}).$$

There is a fiendishly short solution. Fiendish hint: a standard Fibonacci identity is: $F_k F_{k-2} = F_{k-1}^2 + (-1)^{k-1}$. Putting $F_{k-2} = F_k - F_{k-1}$ yields $F_k^2 - F_k F_{k-1} - F_{k-1}^2 = (-1)^{k-1}$.