STERN NOTES, MATH 595, SPRING 2012

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1. Introduction and vague overview

1.1. **Organization.** These notes are about the *Stern sequence*, defined by

$$s(0) = 0, s(1) = 1; \qquad s(2n) = s(n), \quad s(2n+1) = s(n) + s(n+1) \text{ for } n \geq 1. \tag{1.1}$$

This first chapter will introduce the subject of the Stern sequence and related mathematical objects, and give some theorems about them which can be proved directly and quickly. Each subsequent chapter will present a useful mathematical perspective, and then apply it to the topics of the previous chapters. Chapter two is about generating functions and formal power series. Chapter three covers constant-coefficient recurrences and their implications. Chapter four presents some of the classical facts about finite simple continued fractions. Chapter five is devoted to Minkowski's ?-function and related questions in real analysis. Chapters six is about digital representation questions. Chapter seven is about the behavior of $s(n) \pmod{d}$. Chapters eight (and possibly nine) will be about applications and special questions, and I'm not completely sure yet what is likely to show up. I'm taking suggestions.

1.2. **The array and the sequence.** Let's start at the beginning. Stern considered the *diatomic array*, which resembles a Pascal's triangle with memory:

$$\begin{array}{ccccccccc}
 & & & & a \quad b & & & & \\
 & & & a & a+b & b & & & \\
 & & a & 2a+b & a+b & a+2b & b & & \\
a & 3a+b & 2a+b & 3a+2b & a+b & 2a+3b & a+2b & a+3b & b \\
 & & & & \cdots & & & &
\end{array} \tag{1.2}$$

$$\boxed{\boxed{Z(a,b)}}$$

In words, we start with "a, b", and repeat each row in the following row, with the sums of consecutive terms inserted between them. The number of terms in the rows is 2, 3, 5, 9, etc. and is evidently one more than a power of two. So the rows in (1.2) are indexed by r, starting with $r = 0$ and having entries indexed from 0 to 2^r.

The formal definition of $Z(r,k;a,b)$ for $r \ge 0$ and $0 \le k \le 2^r$ is given by:

$$\begin{aligned} Z(0,0;a,b) &= a,\\ Z(0,1;a,b) &= b;\\ Z(r,2k;a,b) &= Z(r-1,k;a,b), \text{ for } r \ge 1;\\ Z(r,2k+1;a,b) &= Z(r-1,k;a,b) + Z(r-1,k+1;a,b), \text{ for } r \ge 1. \end{aligned} \tag{1.3}$$

A few things are evident, and easily proved, from looking at (1.2) and (1.3). First, each entry $Z(r,k;a,b)$ is linear in (a,b) and second, the table has a left-right symmetry if a and b are swapped:

$$Z(r,2^r-k;a,b) = Z(r,k;b,a). \tag{1.4}$$

Application of these two observations allows us to reduce the study of (1.2) to that of a single diatomic array, $Z(r,k;0,1)$:

$$Z(r,k;a,b) = Z(r,k;1,0)a + Z(r,k;0,1)b = Z(r,2^r-k;0,1)a + Z(r,k;0,1)b. \tag{1.5}$$

$$\begin{array}{c} 0 \quad 1 \\ 0 \quad 1 \quad 1 \\ 0 \quad 1 \quad 1 \quad 2 \quad 1 \\ 0 \quad 1 \quad 1 \quad 2 \quad 1 \quad 3 \quad 2 \quad 3 \quad 1 \\ 0 \quad 1 \quad 1 \quad 2 \quad 1 \quad 3 \quad 2 \quad 3 \quad 1 \quad 4 \quad 3 \quad 5 \quad 2 \quad 5 \quad 3 \quad 4 \quad 1 \\ \dots \\ \boxed{\boxed{Z(0,1)}} \end{array} \tag{1.6}$$

Each row of $Z(0,1)$ appears to be repeated as the first half of the next row. This is easily proved by induction and (1.3).

Theorem 1.1. *If $0 \le k \le 2^r$, then $Z(r+1,k;0,1) = Z(r,k;0,1) = s(k)$.*

Proof. Write $Z(r,k;0,1) = Z(r,k)$ for short. By inspection, $Z(1,k) = Z(0,k)$ for $k = 0,1$. Suppose $Z(r_0,k) = Z(r_0-1,k)$ for $0 \le k \le 2^{r_0-1}$. If $k = 2s$ is even and $2s \le 2^{r_0}$, then $s \le 2^{r_0-1}$, and by the induction hypothesis and (1.3), $Z(r_0,2s) = Z(r_0-1,s) = Z(r_0,s) = Z(r_0+1,2s)$. A similar argument holds if $k = 2s+1$ is odd, taking care to check that $2s+1 \le 2^{r_0}$ implies that $s+1 \le 2^{r_0-1}$.

Finally, $Z(r,k) = s(k)$ for small k by inspection, and by (1.1) and (1.3), the two sequences have the same recurrence, so they are always equal. □

(As we shall see in these notes, one must be careful to treat the base cases carefully, since the inductive step is consistent with $Z(r+1,k;a,b) = Z(r,k;a,b)$ for all (a,b), which isn't true.)

It follows from Theorem 1.1 and (1.5) that

$$Z(r,k;a,b) = Z(r,2^r-k;0,1)a + Z(r,k;0,1)b = s(2^r-k)a + s(k)b. \tag{1.7}$$

Another property of $Z(a,b)$ is a kind of self-similarity: any two consecutive elements in a given row can be thought of as the initial conditions of another diatomic array which is generated as a subset of the subsequent rows, contained between the subsequent appearances of the original pair. More formally, if $0 \le r,t$ and $0 \le k \le 2^r, 0 \le n \le 2^t - 1$, then

$$Z(r,k;Z(t,n;a,b),Z(t,n+1;a,b)) = Z(r+t,2^r n+k;a,b). \tag{1.8}$$

(Again, this can be proved by a simple and omitted induction.)

In particular, if $(a,b) = (0,1)$ and $t = \lceil \log_2(n+1) \rceil$ in (1.8), we find that

$$Z(r,k;s(n),s(n+1)) = s(2^r n+k). \tag{1.9}$$

The r-th row below is simply $s(2^r n), s(2^r n+1) \cdots, s(2^r(n+1))$.

$$\begin{array}{ccccc} & & s(n) \quad s(n+1) & & \\ & s(n) & s(n)+s(n+1) & s(n+1) & \\ s(n) \quad 2s(n)+s(n+1) & & s(n)+s(n+1) & & s(n)+2s(n+1) \quad s(n+1) \\ & & \cdots & & \\ & & \boxed{\boxed{Z(s(n),s(n+1))}} & & \end{array} \tag{1.10}$$

Take $n = 1$ in (1.10) and note that $s(1) = s(2) = 1$ so the r-th row consists of the elements $s(2^r), s(2^r+1), \ldots, s(2^{r+1})$. In this way, we can quickly write $s(n)$ for $1 \le n \le 64$:

$$\begin{array}{c} 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 2 \quad 3 \quad 1 \\ 1 \quad 4 \quad 3 \quad 5 \quad 2 \quad 5 \quad 3 \quad 4 \quad 1 \\ 1 \quad 5 \quad 4 \quad 7 \quad 3 \quad 8 \quad 5 \quad 7 \quad 2 \quad 7 \quad 5 \quad 8 \quad 3 \quad 7 \quad 4 \quad 5 \quad 1 \\ 1\ 6\ 5\ 9\ 4\ 11\ 7\ 10\ 3\ 11\ 8\ 13\ 5\ 12\ 7\ 9\ 2\ 9\ 7\ 12\ 5\ 13\ 8\ 11\ 3\ 10\ 7\ 11\ 4\ 9\ 5\ 6\ 1 \\ \cdots \\ \boxed{\boxed{Z(1,1)}} \end{array} \tag{1.11}$$

Note for later reference that, even though $s(2^r)$ appears twice above, as the last entry in the $(r-1)$-st row and the first entry in the r-th row, each *pair* of consecutive entries, $(s(n), s(n+1))$, occurs exactly once for $n \ge 1$: (1,1),(1,2),(2,1),(1,3),(3,2), etc.

By linearity, (1.9) and (1.5) imply that

$$s(2^r n + k) = s(2^r - k)s(n) + s(k)s(n+1) \quad \text{for } 0 \le k \le 2^r. \tag{1.12}$$

This can also be proved directly by induction on r, by considering the parity of k and iterating (1.3). By making the substitutions $(n, k) \mapsto (n-1, 2^r - k)$ in (1.12), we obtain a convenient generalization.

$$s(2^r n \pm k) = s(2^r - k)s(n) + s(k)s(n \pm 1) \quad \text{for } 0 \le k \le 2^r. \tag{1.13}$$

Easy inductions imply that

$$s(2^r) = 1, \qquad s(2^r - 1) = r, \tag{1.14}$$

and so (1.13) implies that

$$s(2^r n \pm 1) = rs(n) + s(n \pm 1). \tag{1.15}$$

For example,

$$\begin{aligned} &s(2012)\\ &= s(1006)\\ &= s(503)\\ &= s(251) + s(252)\\ &= s(125) + 2s(126)\\ &= s(62) + 3s(63)\\ &= 4s(31) + 3s(32)\\ &= 4s(15) + 7s(16)\\ &= 4s(7) + 11s(8)\\ &= 4s(3) + 15s(4)\\ &= 4s(1) + 19s(2)\\ &= 23s(1)\\ &= 23. \end{aligned} \tag{1.16}$$

Notice that the first entry of the argument of s in the r-th row of (1.16) is $\lfloor \frac{2012}{2^r} \rfloor$, and the pattern of whether this argument is even or odd reflects, in reverse order, the binary digits of 2012. See Theorem 1.6 below. Of course, one may stop the computation at any point where $s(n)$ and $s(n+1)$ are known, and use (1.13) to derive these more quickly. For example, $2012 = 64 * 31 + 28$, so $s(2012) = s(31)s(64 - 28) + s(32)s(28)$, and $s(36) = 4, s(28) = 3$, $s(31) = 5$, $s(32) = 1$ imply that $s(2012) = 5 * 4 + 1 * 3 = 23$.

These recurrences also can be used more abstractly. The Stern sequence seems to have many unexpectedly nice properties. For example,

$$\begin{aligned} s((2^r-1)^2) = s(2^{2r}-2^{r+1}+1) = s(2^{r+1}(2^{r-1}-1)+1) = \\ (r+1)s(2^{r-1}-1)+s(2^{r-1}) = (r+1)(r-1)+1 = r^2 = (s(2^r-1))^2. \end{aligned} \tag{1.17}$$

(A check up to 2^{20} shows that $s(n^2) = (s(n))^2$ otherwise for odd n only for $n = 27, 267, 7807$, with no particular pattern evident in the exceptions. This might also be a good place to note that $s(3^n) = 2^n$ for $n = 1, 2, 3, 6$, and for no other $n \le 165$, at least.)

1.3. **Some properties of the rows.** We now turn to some of the properties of the rows of (1.11). Let

$$I_r = \{2^r, 2^r+1, \ldots, 2^{r+1}\}, \tag{1.18}$$

so the r-th row of $Z(1,1)$ consists of $s(n)$ for $n \in I_r$. For $n = 2^r + k \in I_r$, let $n^* = 3 \cdot 2^r - n = 2^{r+1} - k$ denote the reflection of n in I_r. By (1.12),

$$s(n) = s(n^*) = s(k) + s(2^r - k), \qquad s(n^*+1) = s(n-1). \tag{1.19}$$

It is clear that $s(n) \in \mathbb{N}$ and that for $n \ge 1$, $s(2n) < s(2n+1) > s(2n+2)$, so the growth of $s(n)$ will be irregular. Let

$$M_r := \max\{s(n) \ : \ n \in I_r\}. \tag{1.20}$$

An inspection of (1.11) shows that

$$\begin{aligned} M_0 &= s(1) = 1, \\ M_1 &= s(3) = 2, \\ M_2 &= s(5) = s(7) = 3, \\ M_3 &= s(11) = s(13) = 5, \\ M_4 &= s(21) = s(27) = 8, \\ M_5 &= s(43) = s(53) = 13. \end{aligned} \tag{1.21}$$

Let (F_m) denote the usual Fibonacci sequence, defined by $F_0 = 0, F_1 = 1$ and $F_m = F_{m-1} + F_{m-2}$, for $m \ge 2$ and let

$$n_r = \frac{2^{r+2} - (-1)^r}{3} = \frac{4}{3} \cdot 2^r - \frac{(-1)^r}{3}; \quad n_r^* = \frac{5}{3} \cdot 2^r + \frac{(-1)^r}{3}. \tag{1.22}$$

These are the integers closest to $\frac{4}{3} \cdot 2^r$ and $\frac{5}{3} \cdot 2^r$, and effectively trisect I_r.

Theorem 1.2.

$$M_r = s(n_r) = s(n_r^*) = F_{r+2}. \tag{1.23}$$

Proof. The theorem is valid by inspection if $r \le 5$. Suppose $n \in I_r$. If $n = 2k$, then $k \in I_{r-1}$, so $s(n) \le M_{r-1}$. If n is odd, then $n = 4k \pm 1$, and $2k \pm 1 \in I_{r-1}$ and $k \in I_{r-2}$. Thus, $s(n) = s(2k) + s(2k \pm 1) = s(k) + s(2k \pm 1)$, so $s(n) \le M_{r-2} + M_{r-1}$. These arguments imply that $M_r \le M_{r-1} + M_{r-2}$, and, based on the initial conditions, that $M_r \le F_{r+2}$ for all r.

On the other hand, $n_r = 2n_{r-1} - (-1)^r$ and $n_{r-1} - (-1)^r = 2n_{r-2}$, hence

$$s(n_r) = s(n_{r-1}) + s(n_{r-1} - (-1)^r) = s(n_{r-1}) + s(n_{r-2}). \tag{1.24}$$

Since $s(n_r) = F_{r+2}$ for $0 \le r \le 5$, (1.23) follows by induction. As a final remark, if $n \in I_r$ and $s(n) = M_r$, then the argument of the first paragraph implies that $n = 4k \pm 1$, $s(2k \pm 1) = M_{r-1}$ and $s(k) = M_{r-2}$, so these are the only values where the maximum occurs in each row. □

The Binet formula for the Fibonacci numbers states that

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \bar{\phi}^n\right), \qquad \text{where} \quad \phi = \frac{1+\sqrt{5}}{2}, \quad \bar{\phi} = \frac{1-\sqrt{5}}{2}. \tag{1.25}$$

Say that $f(n) = \Theta(g(n))$ if there are positive constants c_j so that $c_1 g(n) \le f(n) \le c_2 g(n)$. If $n \in I_r$, then $\log_2 n - 1 \le r \le \log_2 n$, so $r = \log_2 n + \mathcal{O}(1)$ and

$$\phi^r = e^{r \log \phi} = \Theta(e^{\frac{\log n}{\log 2} \cdot \log \phi}), \tag{1.26}$$

so

$$M_r = \frac{\phi^2}{\sqrt{5}} \cdot \phi^r + o(1) = \Theta(n^\alpha) = o(n), \quad \text{where } \alpha = \frac{\log \phi}{\log 2} \approx .69424. \tag{1.27}$$

By contrast to the irregular growth of $s(n)$, its summatory function is very well-behaved. It is convenient to use a variant notation. For integers $a \le b$, let

$$\begin{aligned} \sum_{n=a}^{b}{}^{*} f(n) &= \sum_{n=a}^{b} f(n) - \tfrac{1}{2}\left(f(a) + f(b)\right) \\ &= \tfrac{1}{2} f(a) + f(a+1) + \cdots + f(b-1) + \tfrac{1}{2} f(b). \end{aligned} \tag{1.28}$$

This is familiar as the trapezoidal estimate to the integral of $f(x)$ from a to b, and has some useful properties

$$\sum_{n=a}^{b}{}^{*} f(n) + \sum_{n=b}^{c}{}^{*} f(n) = \sum_{n=a}^{c}{}^{*} f(n); \qquad \sum_{n=a}^{a}{}^{*} f(n) = 0. \tag{1.29}$$

It will be particularly useful to consider sequences such as

$$\left(\Phi(F(x); r)\right), \quad \text{where } \Phi(F(x); r) := \sum_{n \in I_r}{}^{*} F(s(n)),$$

because $s(2^r) = s(2^{r+1})$ and each $s(2^r)$ would otherwise be counted twice.

Let

$$S(N) := \sum_{n=0}^{N} s(n); \quad S^*(n) := \sum_{n=0}^{N}{}^{*} s(n) = S(N) - \tfrac{1}{2}s(N). \tag{1.30}$$

The following lemma illustrates the utility of this notation.

Lemma 1.3.

$$S^*(2n) = 3S^*(n). \tag{1.31}$$

Proof. We prove a more general result:

$$\begin{aligned}
\sum_{n=2a}^{2b}{}^{*} s(n) &= \tfrac{1}{2}s(2a) + \sum_{k=a+1}^{b-1} s(2k) + \tfrac{1}{2}s(2b) + \sum_{k=a}^{b-1} s(2k+1) \\
&= \tfrac{1}{2}s(a) + \sum_{k=a+1}^{b-1} s(k) + \sum_{k=a}^{b-1}(s(k) + s(k+1)) + \tfrac{1}{2}s(b) \\
&= \tfrac{1}{2}s(a) + \sum_{k=a+1}^{b-1} s(k) + s(a) + \sum_{k=a+1}^{b-1} s(k) + \sum_{k=a+1}^{b-1} s(k) + s(b) + \tfrac{1}{2}s(b) \\
&= 3\left(\tfrac{1}{2}s(a) + \sum_{k=a+1}^{b-1} s(k) + \tfrac{1}{2}s(b)\right) = 3\sum_{n=a}^{b}{}^{*} s(n).
\end{aligned} \tag{1.32}$$

□

Since $S^*(1) = \frac{1}{2}$, it follows that $S^*(2^r) = \frac{1}{2}\cdot 3^r$, $S(2^r) = \frac{1}{2}\cdot(3^r+1)$ and the sum of the r-th row of (1.11) is $S(2^{r+1}) - S(2^r) + 1 = 3^r + 1$, and:

$$\sum_{n=2^r}^{2^{r+1}-1} s(n) = \sum_{n=2^r+1}^{2^{r+1}} s(n) = \sum_{n=2^r}^{2^{r+1}}{}^{*} s(n) = 3^r. \tag{1.33}$$

This means that the average value of $s(n)$ for $n \in I_r$ is roughly $(\frac{3}{2})^r = (2^r)^\beta$, where

$$\beta = \frac{\log \frac{3}{2}}{\log 2} \approx .58496. \tag{1.34}$$

By comparing (1.34) and (1.27), we see that the ratio of the maximum of a row to the average of a row is unbounded, although the ratio grows slowly. For example the maximum value of $s(n)$ for $n \in I_{20}$ is $M_{20} = F_{22} = 17711$, while $(\frac{3}{2})^{20} \approx 3325$.

We shall show later that there is a continuous, strictly increasing function f, mapping $[0,1]$ to itself, with the property that, if $N = 2^r(1+t) \in I_r$, $0 \le t \le 1$, then

$$\frac{1}{3^r}\sum_{n=2^r}^{N}{}^{*} s(n) = f(t). \tag{1.35}$$

That is, the distribution of the "mass" of $s(n), n \in I_r$, is very well-behaved. The function f, which (of course) is differentiable a.e., has the property that $f'(w) = 0$ for every dyadic fraction $w = \frac{p}{2^q}$, but is singular at w when $w = \frac{p}{3 \cdot 2^q}$ is in lowest terms. Note also that by the linearity of the diatomic array and (1.9),

$$(1.36) \quad \sum_{k=0}^{2^r} Z(r,k;a,b) = \frac{3^r+1}{2}(a+b), \qquad \sum_{k=0}^{2^r} s(2^r n + k) = \frac{3^r+1}{2} \cdot (s(n) + s(n+1)).$$

As another example of odd behavior of $s(n)$, it is easy to check that $s(F_r)$ is a Fibonacci number for $1 \le r \le 9$, but not for $10 \le r \le 200$ (at least). Even if we allow Lucas numbers ($L_m = \phi^m + \bar{\phi}^m = F_{m-1} + F_{m+1}$), the only "interesting" hits in this range are $s(L_{15}) = F_{10} = 55$ and $s(F_{27}) = L_{11} = 199$.

1.4. **The consecutive pairs** $(s(n), s(n+1))$**.** It is fair to say that some of the most important properties of the Stern sequence are in fact properties of the pairs $(s(n), s(n+1))$. The first one goes back to Stern himself.

Theorem 1.4. *For $n \ge 0$, $\gcd(s(n), s(n+1)) = 1$. If $a, b \ge 1$ and $\gcd(a,b) = 1$, then there is exactly one n so that $s(n) = a$ and $s(n+1) = b$.*

Proof. Inspection of $Z(1,1)$ shows that $\gcd(s(n), s(n+1)) = 1$ for small n. Since $\gcd(a, a+b) = \gcd(a+b, b) = \gcd(a,b)$, it follows that $s(n)$ and $s(n+1)$ are always relatively prime.

Now suppose $\gcd(a,b) = 1$. We induct on $\max(a,b) = m$. If $m = 1$, then the equation $s(n) = s(n+1) = 1$ clearly holds only for $n = 1$. Assume now that $m \ge 2$ and $m = a > b$. Then $s(n) = a$, $s(n+1) = b$ can only happen if $n = 2n' + 1$ is odd, and only if $s(n') = a - b$ and $s(n'+1) = b$. By the inductive hypothesis, this occurs for exactly one n'. A similar proof can be made if $a < b$, or else we can argue by reflection using (1.19). ☐

We will soon give an alternative proof which constructs n based on the continued fraction expansion of $\frac{a}{b}$.

Corollary 1.5. *Suppose $m \ge 1$. There are exactly $\phi(m)$ odd integers n with the property that $s(n) = m$.*

Proof. If n is odd and $s(n) = m$ and $s(n+1) = k$, then $k < m$ and $\gcd(m,k) = 1$. There are exactly $\phi(m)$ such integers k, and by Theorem 1.4, each k corresponds to exactly one n so that $s(n) = m$ and $s(n+1) = k$. ☐

The series $\sum_n s(n)^{-p}$ is never convergent, because $s(2^r) = 1$ for all r. However, Corollary 1.5 and standard results imply that

$$(1.37) \qquad \sum_{n=0}^{\infty} \frac{1}{(s(2n+1))^p} = \sum_{m=1}^{\infty} \frac{\phi(m)}{m^p} = \frac{\zeta(p-1)}{\zeta(p)},$$

provided $Re(p) > 2$.

Let

$$t(n) = \frac{s(n)}{s(n+1)}. \tag{1.38}$$

It follows from Theorem 1.4 that the sequence $(t(n))$ provides an enumeration of the non-negative rationals, and that one can recover $s(n)$ and $s(n+1)$ unambiguously from $t(n)$. (Cantor was a teenager in 1858, so it's understandable that Stern did not explicitly mention that $\mathbb{Q}$ is countable.) Further, for $n \in I_r$, the mirror symmetry implies that

$$t(n^*-1) = \frac{s(n^*-1)}{s(n^*)} = \frac{s(n+1)}{s(n)} = \frac{1}{t(n)}. \tag{1.39}$$

so reciprocals appear in the same row. We shall see later that

$$\sum_{n=0}^{N} t(n) = \frac{3N}{2} + \mathcal{O}((\log N)^2). \tag{1.40}$$

One can then argue that the "average" positive rational number is $\frac{3}{2}$.

There are several natural ways to express the sequence $(t(n))$. Historically, the first was found by a French watchmaker named Achille Brocot, independently of Stern's work, who was interested in making a practical table of "gear ratios". His table was computed by starting with the fractions $\frac{0}{1}, \frac{1}{0}$, and then, if $\frac{a}{b}, \frac{c}{d}$ are consecutive in the r-th row, they are repeated in the $r+1$-st row, with $\frac{a+c}{b+d}$ inserted between them:

$$\begin{array}{ccccccccc}
 & & & \frac{0}{1} & & \frac{1}{0} & & & \\
 & & & \frac{0}{1} & \frac{1}{1} & \frac{1}{0} & & & \\
 & & \frac{0}{1} & \frac{1}{2} & \frac{1}{1} & \frac{2}{1} & \frac{1}{0} & & \\
\frac{0}{1} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{1}{1} & \frac{3}{2} & \frac{2}{1} & \frac{3}{1} & \frac{1}{0} \\
 & & & & \cdots & & & &
\end{array} \tag{1.41}$$

Brocot array

In Stern terminology, the r-th row of the Borcot array consists of $\frac{s(k)}{s(2^r-k)}$; the numerators are the r-th row of $Z(0,1)$, the denominators are its reversal, or the r-th row of $Z(1,0)$. Each row is increasing from left-to-right. It is not immediately clear how to decode $t(n)$ from this array.

More directly, write the elements of $t(n)$ which appear in each row.

$$\begin{array}{cccccccc} & & & & \frac{1}{1} & & & \\ & & & \frac{1}{2} & \frac{2}{1} & & & \\ & & \frac{1}{3} & \frac{3}{2} & \frac{2}{3} & \frac{3}{1} & & \\ \frac{1}{4} & \frac{4}{3} & \frac{3}{5} & \frac{5}{2} & \frac{2}{5} & \frac{5}{3} & \frac{3}{4} & \frac{4}{1} \\ & & & & \cdots & & & \end{array} \tag{1.42}$$

$\boxed{t(n)}$

There is a subtle and confusing correspondence between these two arrays. The odd elements in the r-th row of the Brocot array, $\frac{s(2k+1)}{s(2^r-(2k+1))}$, appear in the $(r-1)$-st row of the $t(n)$ array, in a different order. This depends on reversing binary representations in a way that will be described later. Note that the rows of the $t(n)$ array are *not* constructed as simply as Farey sequences: $\frac{2}{5}$ and $\frac{3}{5}$ appear in the last row, but $\frac{1}{5}$ and $\frac{4}{5}$ do not.

Interestingly, the Stern sequence provides a second enumeration of the rationals. If $\gcd(a,b)=1$, not only is there a unique n so that $\frac{a}{b}=t(n)$, but there is also a unique odd k and r so that $\frac{a}{b}=\frac{s(k)}{s(2^r+k)}$. The array whose r-th row is $\frac{s(k)}{s(2^r+k)}$ has monotone rows, and is essential to Minkow- ski's bizarrely-named ?-function. You may recognize it as a Brocot array starting with $\frac{0}{1},\frac{1}{1}$ instead of $\frac{0}{1},\frac{1}{0}$; the k-th element in the r-th row is $\frac{s(k)}{s(2^r+k)}$:

$$\begin{array}{ccccccccc} & & & \frac{0}{1} & & \frac{1}{1} & & & \\ & & & \frac{0}{1} & \frac{1}{2} & \frac{1}{1} & & & \\ & & \frac{0}{1} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{1}{1} & & \\ \frac{0}{1} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{1}{1} \\ & & & & \cdots & & & & \end{array} \tag{1.43}$$

$\boxed{\text{Minkowski ?- array}}$

The *Minkowski ?-function*, which maps $[0,1]$ to itself, is defined recursively by taking the initial conditions $?(0)=0$, $?(1)=1$, and recursively applying the rule:

$$?\left(\tfrac{a}{b}\right)=\tfrac{k}{2^r},\ ?\left(\tfrac{c}{d}\right)=\tfrac{k+1}{2^r}\implies ?\left(\tfrac{a+c}{b+d}\right)=\tfrac{2k+1}{2^{r+1}}. \tag{1.44}$$

(We assume above that $\frac{a}{b}, \frac{c}{d}$ are given in lowest terms.) This definition only gives $?(x)$ for rational $x \in [0,1]$: $?(\frac{s(k)}{s(2^r+k)}) = \frac{k}{2^r}$, for $0 \le k \le 2^r$, but we shall see that ? extends uniquely to a continuous function.

1.5. **Continued fractions make their presence known.** There is a simple closed form for $t(n)$, which leads to simple formula for $s(n)$. This formula is useful for many symbolic purposes; however, for any specific numerical n, it is usually easier to use (1.1) directly to compute n; c.f. (1.16).

Suppose n is a positive integer, so $n \in I_r$ for some r, and let $[n]_2$ denote the usual binary representation of n, reading left-to-right, with no leading 0's and first digit 1:

$$n = \sum_{j=0}^{r} \epsilon_j(n) 2^j, \quad \epsilon_j(n) \in \{0,1\} \implies [n]_2 = [\epsilon_r(n), \dots, \epsilon_0(n)]_2 \tag{1.45}$$

It is useful to think of $[n]_2$ in terms of the blocks of consecutive 0's and 1's, and it is also useful to distinguish by the parity of n. If n is odd, then $[n]_2$ consists of an odd number of alternating blocks of 1's and 0's, beginning and ending with 1's. More specifically, suppose $[n]_2$ has a_1 1's, followed by a_2 0's, a_3 1's, etc, ending with a_{2v} 0's and a_{2v+1} 1's. (By convention, assume that $a_j \ge 1$.) We write $n \sim [a_1, \dots, a_{2v+1}]$ and observe that

$$n = 2^{a_1+\cdots+a_{2v+1}} - 2^{a_2+\cdots+a_{2v+1}} + - \cdots + 2^{a_{2v+1}} - 1. \tag{1.46}$$

If n is even, then $[n]_2$ consists of an even number of alternating blocks, beginning with 1's and ending with 0's. If $[n]_2$ has a_1 1's, followed by a_2 0's, a_3 1's, etc, ending with a_{2v-1} 1's and a_{2v} 0's, we write $n \sim [a_1, \dots, a_{2v}]$ and observe that

$$n = 2^{a_1+\cdots+a_{2v}} - 2^{a_2+\cdots+a_{2v}} + - \cdots + 2^{a_{2v-1}+a_{2v}} - 2^{a_{2v}}. \tag{1.47}$$

Suppose $n \sim [a_1, \dots, a_u]$ and $n' \sim [a_1, \dots, a_{u-1}]$. If n is odd, then n' is even, and $n = 2^{a_u} n' + 2^{a_u} - 1$; if n is even, then n' is odd, and $n = 2^{a_u} n'$. In either case, $\sum_j a_j = r + 1$. For example,

$$\begin{gathered} 243 = 2^7 + 2^6 + 2^5 + 2^4 + 2^1 + 2^0 = [11110011]_2 \implies 243 \sim [4,2,2], \\ 140 = 2^7 + 2^3 + 2^2 = [10001100]_2 \implies 140 \sim [1,3,2,2] \end{gathered} \tag{1.48}$$

Consider (1.42) as a binary tree, with the nodes labeled by the consecutive integers.

$$\begin{array}{ccccccc} & & & [1]_2 & & & \\ & & \swarrow & & \searrow & & \\ & [10]_2 & & & & [11]_2 & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ [100]_2 & & [101]_2 & & [110]_2 & & [111]_2 \\ & & & \cdots & & & \end{array} \tag{1.49}$$

Now write the binary tree with $t(n)$ at the node labeled by $[n]_2$.

$$
\begin{array}{ccccccc}
 & & & \frac{1}{1} & & & \\
 & & \swarrow & & \searrow & & \\
 & \frac{1}{2} & & & & \frac{2}{1} & \\
\swarrow & & \searrow & & \swarrow & & \searrow \\
\frac{1}{3} & & \frac{3}{2} & & \frac{2}{3} & & \frac{3}{1} \\
 & & & \cdots & & &
\end{array}
\tag{1.50}
$$

From the definition,

$$
\begin{aligned}
t(2n) &= \frac{s(2n)}{s(2n+1)} = \frac{s(n)}{s(n)+s(n+1)} = \frac{1}{1+\frac{1}{t(n)}} = \frac{t(n)}{t(n)+1} \\
t(2n+1) &= \frac{s(2n+1)}{s(2n+2)} = \frac{s(n)+s(n+1)}{s(n+1)} = t(n)+1.
\end{aligned}
\tag{1.51}
$$

Let

$$
f_0(x) = \frac{x}{x+1}, \qquad f_1(x) = x+1. \tag{1.52}
$$

Then we see that, using the terminology of (1.45) and noting that $\epsilon_r(n) = 1$ for all n and $t(0) = 0$, we have

$$
t(n) = f_{\epsilon_0(n)}(f_{\epsilon_1(n)}(\cdots(f_{\epsilon_{r-1}(n)}(f_{\epsilon_r(n)}(0)))\cdots)). \tag{1.53}
$$

If we let $g^{(k)}$ denote the k-th iterate of a function g, it is routine to check that

$$
f_0^{(k)}(x) = \frac{x}{kx+1} = \frac{1}{k+\frac{1}{x}}, \qquad f_1^k(x) = x+k. \tag{1.54}
$$

The effect of appending k 0's or k 1's to $[n]_2$ implies that

$$
t(2^k n) = \frac{1}{k+\frac{1}{t(n)}}, \qquad t(2^k n + 2^k - 1) = k + t(n). \tag{1.55}
$$

Also note that similar expressions exist if one chooses to view the consecutive pair of Stern-values as a column matrix rather than as a fraction. Let

$$
M_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{1.56}
$$

Then

$$
\begin{bmatrix} s(2n) \\ s(2n+1) \end{bmatrix} = M_0 \begin{bmatrix} s(n) \\ s(n+1) \end{bmatrix}, \qquad \begin{bmatrix} s(2n+1) \\ s(2n+2) \end{bmatrix} = M_1 \begin{bmatrix} s(n) \\ s(n+1) \end{bmatrix}. \tag{1.57}
$$

It follows that

$$
\begin{bmatrix} s(n) \\ s(n+1) \end{bmatrix} = M_{\epsilon_0(n)} M_{\epsilon_1(n)} \cdots M_{\epsilon_{r-1}(n)} M_{\epsilon_r(n)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1.58}
$$

The powers of these matrices are familiar from the study of continued fractions.

$$M_0^k = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}, \qquad M_1^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}. \tag{1.59}$$

Theorem 1.6.

(i) If n is odd and $n \sim [a_1, \dots, a_{2v+1}]$, then

$$t(n) = \frac{s(n)}{s(n+1)} = a_{2v+1} + \frac{1}{a_{2v} + \frac{1}{\dots + \frac{1}{a_1}}}. \tag{1.60}$$

(ii) If n is even and $n \sim [a_1, \dots, a_{2v}]$, then

$$t(n) = \frac{s(n)}{s(n+1)} = \frac{1}{a_{2v} + \frac{1}{\dots + \frac{1}{a_1}}}. \tag{1.61}$$

Proof. Let $n \sim [a_1, \dots, a_k]$. We argue by induction on k. If $k = 1$, then $n \sim [a_1]$, so $n = 2^{a_1} - 1$ and (1.14) shows that

$$t(2^r - 1) = \frac{s(2^r - 1)}{s(2^r)} = \frac{r}{1} = r. \tag{1.62}$$

If $n > 1$ is odd, then k is odd, $n = 2^{a_{2v+1}} n' + 2^{a_{2v+1}} - 1$, where n' is even, and (1.61) applies to n' by the inductive hypothesis. By (1.55), (1.60) holds for n. Similarly, if $n > 0$ is even, then k is even, $n = 2^{a_{2v+1}} n'$ where n' is odd, and (1.60) applies to n' by the inductive hypothesis. By (1.55), (1.61) holds for n. □

Suppose a rational number $\frac{a}{b} > 1$ is given. One may write it as a simple continued fraction (i.e., as in (1.60)) in two ways, because the final denominator may either be chosen as m (for $m \ge 2$) or $(m-1) + \frac{1}{1}$. Exactly one of these representations will have an odd number of denominators, to which Theorem 1.6 will apply, stating that $s(n) = a, s(n+1) = b$. If $\frac{a}{b} < 1$, either apply Theorem 1.6 to $\frac{b}{a}$ and apply (1.39), or use the representation of $\frac{b}{a}$ with an even number of denominators and apply Theorem 1.6 directly.

Later on, we'll see that Theorem 1.6 implies that the ?-function gives a strictly increasing bijection of the rationals in $[0,1]$ onto the dyadic rationals in $[0,1]$. The properties of periodic infinite continued fractions imply that that the ?-function gives a strictly increasing bijection of the algebraic numbers of degree ≤ 2 in $[0,1]$ onto the rationals.

Further, $h(a,b)$, the sum of the denominators in the continued fraction representation of $\frac{a}{b}$ does not depend on which choice of representation is used, and equals the number of binary digits in n, so $n \in I_{h(a,b)-1}$ and $\frac{a}{b}$ will appear in row $h(a,b) - 1$. For example, $4 = 2+1+1 = 1+2+1 = 1+1+2$ are the ordered partitions of 4 as an odd number of summands and

$$4 = \frac{4}{1}, \quad 2 + \frac{1}{1 + \frac{1}{1}} = \frac{5}{2}, \quad 1 + \frac{1}{2 + \frac{1}{1}} = \frac{4}{3}, \quad 1 + \frac{1}{1 + \frac{1}{2}} = \frac{5}{3}. \tag{1.63}$$

These (together with their reciprocals) are the entries of the fourth row of (1.42).

One somewhat surprising consequence of the properties of simple continued fractions is the following. For odd n, let $\overleftarrow{n}$ denote the integer whose binary expression is the reversal of n; more formally, if $n \sim [a_1, \dots, a_{2v+1}]$, then $\overleftarrow{n} \sim [a_{2v+1}, \dots, a_1]$. It turns out that $s(\overleftarrow{n}) = s(n)$ and $s(n+1)s(\overleftarrow{n}+1) \equiv 1 \pmod{s(n)}$. It also turns out that $[n^*]_2$ can be expressed in terms of $[n]_2$ and that $\overleftarrow{(n^*)} = (\overleftarrow{n})^*$. Indeed, suppose (1.45) holds. Then we first claim that

$$n^* = 1 + \sum_{j=1}^{r-1} (1 - \epsilon_j(n)) 2^j + 2^r, \tag{1.64}$$

because this is equivalent to the assertion that $n + n^* = 2 + \sum_{j=1}^{r-1} 2^j + 2 \cdot 2^r = 3 \cdot 2^r$. Similarly, we obtain

$$\overleftarrow{n} = 1 + \sum_{j=1}^{r-1} \epsilon_{r-j}(n) 2^j + 2^r, \quad \overleftarrow{(n^*)} = (\overleftarrow{n})^* = 1 + \sum_{j=1}^{r-1} (1 - \epsilon_{r-j}(n)) 2^j + 2^r. \tag{1.65}$$

Suppose n is odd and

$$s(n-1) = m - a, \quad s(n) = m, \quad s(n+1) = a. \tag{1.66}$$

Then we have

$$s(n^* - 1) = a, \quad s(n^*) = m, \quad s(n+1) = m - a, \tag{1.67}$$

and if $b < m$ is such that $ab \equiv 1 \pmod m$, then (after a later proof),

$$\begin{gathered} s(\overleftarrow{n} - 1) = m - b, \quad s(\overleftarrow{n}) = m, \quad s(\overleftarrow{n} + 1) = b \\ s(\overleftarrow{n}^* - 1) = b, \quad s(\overleftarrow{n}^*) = m, \quad s(\overleftarrow{n}^* + 1) = m - b. \end{gathered} \tag{1.68}$$

Unless $n = 3$, it is always the case that $n \neq n^*$, although $n = \overleftarrow{n}$ or $n = \overleftarrow{n}^*$ occur roughly $2^{r/2}$ times in the r-th row; these correspond to $a^2 \equiv \pm 1 \pmod m$, where $m = s(n)$. Thus one can usually expect $s(n) = m$ to occur in groups of four odd n in a row for a given m – for $n, n^*, \overleftarrow{n}, \overleftarrow{n^*}$.

As a numerical example, using the computation $243 \sim [4, 2, 2]$ from (1.48), we have

$$t(243) = \frac{s(243)}{s(244)} = 2 + \frac{1}{2 + \frac{1}{4}} = 2 + \frac{4}{9} = \frac{22}{9}, \tag{1.69}$$

so $s(243) = 22$ and $s(244) = 9$. As a double-check of reflection, $244^* = 140$, $243^* = 141$ and $140 \sim [1, 3, 2, 2]$ so

$$t(140) = \frac{s(140)}{s(141)} = \frac{1}{2 + \frac{1}{2 + \frac{1}{3 + \frac{1}{1}}}} = \frac{1}{2 + \frac{1}{2 + \frac{1}{4}}} = \frac{9}{22}. \tag{1.70}$$

Since $[\overleftarrow{243}]_2 = 11001111$, $\overleftarrow{243} = 207$, and

$$t(207) = \frac{s(207)}{s(208)} = 4 + \frac{1}{2+\frac{1}{2}} = 4 + \frac{2}{5} = \frac{22}{5}. \tag{1.71}$$

Note that $9 \cdot 5 = 22 \cdot 2 + 1$, that $243^* = 3 \cdot 128 - 243 = 141$, $[141]_2 = 10001101$, $[\overleftarrow{141}]_2 = 10110001$, so $\overleftarrow{141} = 177 = 207^*$, and just to finish up,

$$t(141) = \frac{s(141)}{s(142)} = \frac{22}{13}, \quad t(177) = \frac{s(177)}{s(178)} = \frac{22}{17}. \tag{1.72}$$

We can use Theorem 1.6 to determine the solutions to the equation $s(n) = m$ for fixed m. For example, suppose n is odd and $s(n) = 12$. Since $\gcd(12, s(n+1)) = 1$ and $s(n+1) < 12$, we must have $s(n+1) \in \{1, 5, 7, 11\}$. Every positive rational has a finite continued fraction: and, as

$$\frac{12}{1} = 12, \quad \frac{12}{5} = 2 + \frac{1}{2+\frac{1}{2}}, \quad \frac{12}{7} = 1 + \frac{1}{1+\frac{1}{2+\frac{1}{2}}}, \quad \frac{12}{11} = 1 + \frac{1}{11}. \tag{1.73}$$

Since the last two expressions have an even number of denominators, we tweak the innermost denominator to give an odd length:

$$\frac{12}{7} = 1 + \frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{1}}}}, \qquad \frac{12}{11} = 1 + \frac{1}{10+\frac{1}{1}}. \tag{1.74}$$

Thus, we see that $s(n) = 12$ when $[n]_2 \sim [12], [2,2,2], [1,1,2,1,1]$ or $[1,10,1]$, that is, when $[n]_2 = [111111111111]_2 = 2^{12} - 1 = 4095$, $[n]_2 = [110011]_2 = 51$, $[n]_2 = [101101]_2 = 45$ or $[n]_2 = [100000000001]_2 = 2^{11} + 1 = 2049$. Note that $45^* = 51$ and $2049^* = 4093$, and all four binary expressions are palindromes, since $1^2, 5^2, 7^2, 11^2 \equiv 1 \pmod{24}$.

As a symbolic example, $(2^r - 1)^2 = 2^{2r} - 2^{r+1} + 1 = 2^{2r-1} + 2^{2r-2} + \cdots + 2^{r+1} + 1$, so $(2^r - 1)^2 \sim [r-1, r, 1]$ and,

$$t((2^r-1)^2) = \frac{s((2^r-1)^2)}{s((2^r-1)^2+1)} = 1 + \frac{1}{r+\frac{1}{r-1}} = \frac{r^2}{r^2-r+1}. \tag{1.75}$$

1.6. **The Stern sequence mod d and some combinatorial interpretations.** For integers $d \geq 2$ and $0 \leq i \leq d-1$, let

$$A(d,i) = \{n : s(n) \equiv i \pmod d\}, \tag{1.76}$$

and let

$$\begin{aligned} T(N;d,i) &= |\{n \in A(d,i) : 0 \leq n < N\}|, \\ U(r;d,i) &= T(2^{r+1};d,i) - T(2^r;d,i). \end{aligned} \tag{1.77}$$

Stern noted that $s(n) \pmod 2$ is periodic with period 3; in this notation, $A(2;0) = 3\mathbb{N}$ and $T(N;2,0) = \lfloor \frac{n+2}{3} \rfloor$, so $U(r;2,0) = \frac{1}{3}(2^r - (-1)^r)$. We shall prove later that each $U(r;d,i)$ satisfies a linear recurrence, and

$$u(d;i) := \lim_{r\to\infty} \frac{U(r;d,i)}{2^r} \tag{1.78}$$

exists as a computable arithmetic function of (d,i). A stronger statement is true:

$$\lim_{N\to\infty} \frac{T(N;d,i)}{N} = u(d,i); \tag{1.79}$$

that is, the limit applies even through elements in the middle of the rows.

Define the arithmetic function $I(m)$ by

$$\frac{I(m)}{m} = \prod_{p\ \mid\ m} \frac{p+1}{p}. \tag{1.80}$$

Then $u(d,0) = \frac{1}{I(d)}$. In some cases, when $I(d_1) = I(d_2)$, an even stronger statement can be made: $I(4) = I(5) = 6$ and $U(r;4;0) = U(r;5,0)$. That is the number of multiples of 4 and the number of multiples of 5 is the same for $\{s(n) : n \in I_r\}$. Further, $I(6) = I(8) = I(9) = I(11) = 12$, and $U(r;6,0) = U(r,9,0) = U(r;11,0)$, but $U(8;r,0)$ is different. The function I has interesting iterative behavior: there exist $a(d), b(d)$ such that, for each integer d and sufficiently large N, $I^{(N)}(d) = 2^{N+a(d)}3^{b(d)}$.

These observations are a consequence of studying the behavior of Stern pairs modulo d. Note that if $(s(n), s(n+1)) \equiv (a,b) \pmod d$, then $\gcd(a,b,d) = 1$. It turns out that the residue classes are always uniformly distributed among these possible pairs. The argument requires a Markov chain model.

In case $d = 3$, stronger information can be presented. Define the set $\mathcal{S}_3 \subset \mathbb{N}$ recursively by:

$$0, 5, 7 \in \mathcal{S}_3, \qquad 0 < n \in \mathcal{S}_3 \implies 2n, 8n \pm 5, 8n \pm 7 \in \mathcal{S}_3. \tag{1.81}$$

(Thus, the smallest non-negative integers in $\mathcal{S}_3$ are: 0, 5, 7, 10, 14, 20, 28, 33, 35, 40, 45, 47, 49, 51, 56, 61, 63.) This is a member of an interesting family of recursively defined sets, and associated directed graphs on $\mathbb{Z}$.

Theorem 1.7. *$A(3,0) = \mathcal{S}_3$.*

Proof. We first observe that by (1.13),

$$s(2n) = s(n), \quad s(8n\pm 5) = 2s(n)+3s(n\pm 1), \qquad s(8n\pm 7) = s(n)+3s(n\pm 1). \tag{1.82}$$

Thus, 3 divides $s(n)$ if and only if 3 divides $s(2n), s(8n \pm 5), s(8n \pm 7)$. Every $n \in \mathbb{N}$ belongs to exactly one of the congruence classes $0 \pmod 2$, $\pm 5 \pmod 8$, $\pm 7 \pmod 8$, and if $n \geq 2$, then $n = 2n', 8n' \pm 5$ or $8n' \pm 7$ with $n' < n$. Thus, the inductive construction of $\mathcal{S}_3$ gives all n for which $s(n)$ is a multiple of 3. ☐

We shall also show that $T(N;3,0) = \frac{1}{4}N + \mathcal{O}(N^{1/2})$, with the error bound best possible, and $T(N;3,1) - T(N;3,2) \in \{0,1,2,3\}$.

We turn to digital questions. Let $\mathcal{A} = \{0 = a_0 < a_1 < \cdots a_r\}$ denote a finite subset of $\mathbb{N}$ containing 0, and let $f_{\mathcal{A}}(m)$ denote the number of ways to write m in the form

$$m = \sum_{k=0}^{\infty} \epsilon_k 2^k, \qquad \epsilon_k \in \mathcal{A}. \tag{1.83}$$

For example, the binary representation of m implies that $f_{\{0,1\}}(m) = 1$ for all m.

Theorem 1.8.

$$s(n) = f_{\{0,1,2\}}(n-1). \tag{1.84}$$

Proof. Let $f(m) = f_{\{0,1,2\}}(m)$ for short. Observe that $0 = s(0) = f(-1)$ trivially and $1 = s(1) = f(0)$, since the only way to write 0 as (1.83) is to have $\epsilon_k \equiv 0$ for all k. We now show that

$$f(2n-1) = f(n-1), \qquad f(2n) = f(n) + f(n-1), \tag{1.85}$$

and this will establish the theorem by induction, by comparison with (1.1) and (1.84)

Notice that $m \equiv \epsilon_0 \pmod 2$ in (1.83); moreover,

$$m = \epsilon_0 + 2\sum_{j=0}^{\infty} \epsilon_{j+1} 2^j = \epsilon_0 + 2m', \tag{1.86}$$

and the representation of m' obeys (1.83) as well. It follows that in any representation of $2n-1$ in (1.83), we must have $\epsilon_0 = 1$, with $m' = n-1$, and in any representation of $2n$ in (1.83), we may have $\epsilon_0 = 0$ or 2, with $m = n$ or $n-1$ respectively. □

Theorem 1.8 will become more transparent when we discuss the generating function for the Stern sequence:

$$\sum_{n=0}^{\infty} s(n)x^n = x\prod_{j=0}^{\infty}(1 + x^{2^j} + x^{2^{j+1}}).$$

Another combinatorial interpretation is almost trivial. Let G be a directed graph whose vertices are $\mathbb{N}$ and whose directed edges are precisely those of the the form $(2k, k), (2k+1, k), (2k+1, k+1)$. Then $s(n)$ is the number of paths from n to 1. Somewhat surprisingly, G is planar: to draw it in this way, put the vertices from I_r, $n = 2^r + k$, at the points $(\log(n+1))e^{2\pi i \frac{k}{2^r}}$ on a spiral in the plane.

1.7. **Exercises.** Do some of these. Extensions of Problem 10 will show up in subsequent exercise sets, so it's a good one to do. Let's say a deadline of Fri. Feb. 3, at the beginning of class, at which point I'll pass out solutions.

1. Write today's date as MMDDYYYY $\in [10^7, 10^9)$ and compute $s(n)$. For example, the first day of class was Jan. 18, 2012, and $s(01182012) = 1244$. (Europeans should use DDMMYYYY; $s(18012012) = 15394$.) This problem can be done on several days in a row, especially using a program.

2. Determine n so that $s(n) = 2012$ and $s(n+1) = 595$. Note that $2012 = 2^2 \cdot 503$ and $595 = 5 \cdot 7 \cdot 17$ are relatively prime.

3. Prove that

$$\sum_{n=0}^{N}{}^{*} n = \frac{N^2}{2}, \qquad \sum_{n=0}^{N}{}^{*} n^2 = \frac{N^3}{3} + \frac{N}{6},$$

and compute

$$\sum_{n=0}^{N}{}^{*} n^3.$$

4. Let $\nu_p(n)$ denote the exponent of p in the prime factorization of n. Show that

$$\frac{s(n-1) + s(n+1)}{s(n)} = 1 + 2\nu_2(n).$$

5. Determine, by any correct method, all odd integers n so that $s(n) \in \{10, 11\}$.

6. Using (1.22), compute $[n_r]_2$; there are two slightly different answers, depending on the parity of r.

7. Find and prove a formula relating

$$T(n) := \sum_{k=0}^{\lfloor n/3 \rfloor} s(n - 3k)$$

and $S(n)$.

8. For $r \ge 1$ and $t \ge 0$, compute $s((2^r + 1)^2)$ and $s((2^r - 1)(2^{r+t} - 1))$.

9. Sometimes the Stern sequence fakes you out. Suppose

$$c_r = \left(\sum_{n \in I_r}{}^{*} 1\right)\left(\sum_{n \in I_r}{}^{*} s(n)^2\right) - \left(\sum_{n \in I_r} s(n)\right)^2.$$

We know that $c_r \ge 0$ by the Cauchy-Schwarz Inequatilty. Show that $c_1 = 1$, $c_2 = 11$ and $c_3 = 111$. Compute the disheartening value of c_4.

10. (The first in a series.) Show that for $k \in \mathbb{N}$, there exist functions $A(k), B(k)$ so that for $r > \log_2 k$,

$$s(2^r - k) = A(k)r + B(k).$$

The most instructive way to do this problem is to see what happens for small values of k first. The recursion is helpful, the continued fraction, less so.