PICK'S THEOREM REVISITED

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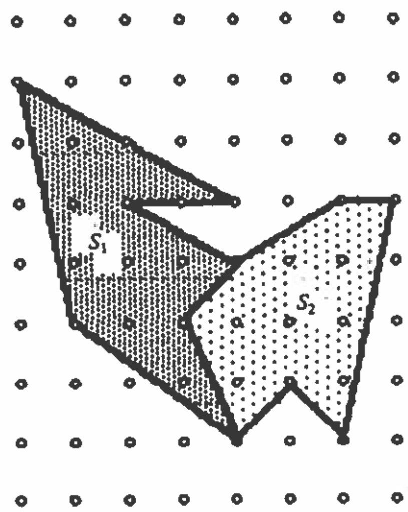
Pick's theorem is one of the gems of elementary mathematics, this because the most innocent sounding hypotheses imply a very surprising conclusion. Yet the statement of the theorem can be understood by a fifth grader. Call a polygon simple if its boundary is a simple closed curve and call it a lattice polygon if the coordinates of its vertices are integers. Pick's theorem asserts that the area of a simple lattice polygon S is given by

$$A(S) = i + \frac{1}{2}b - 1 = v - \frac{1}{2}b - 1$$

where i, b, and v are, respectively, the number of interior lattice points, the number of boundary lattice points, and the total number of lattice points of S (Fig. 1).



$$A(S) = i + \frac{b}{2} - 1$$

$$A(S_1) = 7 + \tfrac{9}{2} - 1 = \tfrac{21}{2}$$

$$A(S_2) = 7 + \tfrac{7}{2} - 1 = \tfrac{19}{2}$$

$$A(S_1 \cup S_2) = 15 + \tfrac{12}{2} - 1 = 20$$

FIG. 1

We are aware of several proofs of Pick's theorem [1], [2], [4], [5], [6]. Each of them uses an elegant trick or two but finally rests on the stubborn fact that a primitive triangle (one whose only lattice points are its vertices) has area 1/2. We call this a stubborn fact because its derivation seems to be the major hurdle in all the standard proofs.

Our proof of Pick's theorem is direct, intuitive, and requires nothing more sophisticated than the almost obvious and easily proved fact (proof by induction will work) that a lattice polygon can be decomposed as a union of lattice triangles.

To begin, associate with each lattice point P_k of S a weight $w_k = \theta_k/2\pi$, where θ_k measures the "visibility" angle with which P_k can see into S. Thus $w_k = 1$ at an interior lattice point, $w_k = 1/2$ at a boundary lattice point that is not a vertex, and $w_k = 1/4$ at a right-angled corner point. Think of w_k as measuring the contribution that P_k makes to the area of S (see Fig. 2). Let

$$W(S) = \sum_{P_k \in S} w_k$$

and, as before, let $A(S)$ be the area of S.

LEMMA. $W(S) = A(S)$.

Proof. We note first that W is additive; that is, if $S = S_1 \cup S_2$ as in Fig. 1, then $W(S) =$

$W(S_1) + W(S_2)$. This is a consequence of the fact that the visibility angles in S_1 and S_2 at a common lattice point add together to give the visibility angle in S at that point.

Next consider successively (i) a lattice rectangle with sides parallel to the lattice (Fig. 2), (ii) a lattice right triangle with legs parallel to the lattice (Fig. 3), and (iii) an arbitrary lattice triangle. The latter can be surrounded by right triangles of type (ii) to form a rectangle of type (i) (Fig. 4 shows a typical example).

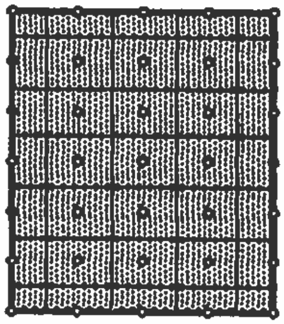


FIG. 2

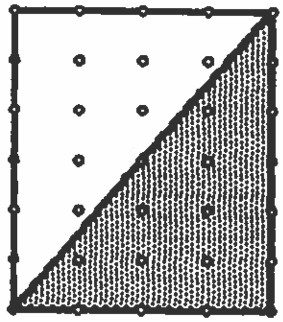


FIG. 3

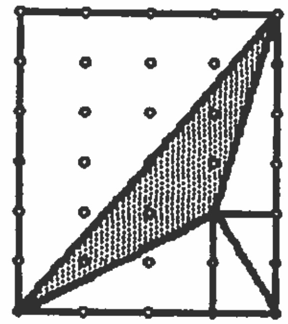


FIG. 4

That $W(S) = A(S)$ in Case (i) is obvious and Case (ii) follows immediately from Case (i) upon division by 2. We obtain the result in Case (iii) by using the shared additivity of W and A. It remains only to decompose an arbitrary lattice polygon as a union of lattice triangles and then to apply the additivity of W.

We remark that the simpleness of S is not used in the proof of the lemma. ■

PICK'S THEOREM. *For a simple lattice polygon,*

$$A(S) = i + \frac{1}{2}b - 1 = v - \frac{1}{2}b - 1.$$

Proof. A simple polygon with c interior vertex angles has angle sum $(c-2)\pi$. For example, a triangle has angle sum π, a quadrilateral 2π, and so on. It follows that the sum of all the visibility angles at points P_k along the boundary of S is $(b-2)\pi$. Thus if I and B denote the interior and boundary of S, then

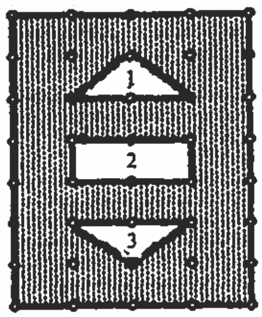
$$\begin{aligned} A(S) = W(S) &= \sum_{P_k \in I} w_k + \sum_{P_k \in B} w_k \\ &= i + \frac{(b-2)\pi}{2\pi} \\ &= i + \frac{1}{2}b - 1. \blacksquare \end{aligned}$$

Generalizations. Consider next a general lattice polygon S, one that may have holes or intersect itself (see Figs. 5, 6, and 7). We require only that S can be expressed as the union of finitely many simple lattice polygons. Pick's theorem as originally stated does not apply to such figures but a simple variant involving the Euler characteristic χ does. The correct formula is

$$A(S) = v - \frac{1}{2}e_b - \chi.$$

Here, e_b is the number of boundary edges of S. Note that for a simple lattice polygon, $e_b = b$ and $\chi = 1$.

The situation illustrated in Fig. 5 is easily handled with no additional apparatus. Let S be a simple lattice polygon containing m separated holes which are themselves simple lattice polygons. Recall that the Euler characteristic for such a figure is $1 - m$. Let $b_0, b_1, b_2, \ldots, b_m$ denote the number of lattice points along the outer boundary of S and the boundaries of the m holes of S,

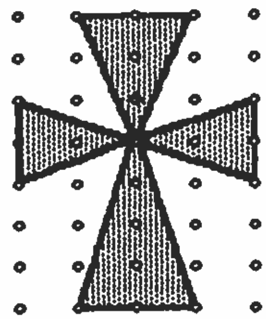


$v = 40 \qquad e_b = 36$

$\chi = -2$

$A(S) = 40 - \frac{1}{2}(36) + 2$

FIG. 5

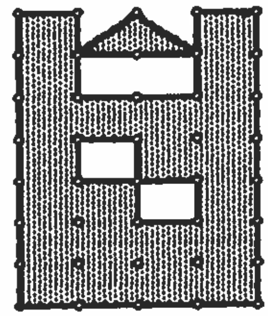


$v = 20 \qquad e_b = 16$

$\chi = 1$

$A(S) = 20 - \frac{1}{2}(16) - 1$

FIG. 6



$v = 40 \qquad e_b = 38$

$\chi = -2$

$A(S) = 40 - \frac{1}{2}(38) + 2$

FIG. 7

respectively. Then, using the fact that the visibility angles of the kth hole sum to $(b_k + 2)\pi$, we obtain

$$\begin{aligned} A(S) &= i + \frac{1}{2\pi} \sum_{P_k \in B} \theta_k \\ &= i + \frac{(b_0 - 2)}{2} + \frac{(b_1 + 2)}{2} + \cdots + \frac{(b_m + 2)}{2} \\ &= i + \frac{1}{2}(b_0 + b_1 + \cdots + b_m) + (m - 1) \\ &= i + \frac{1}{2}b - \chi = v - \frac{1}{2}b - \chi. \end{aligned}$$

This demonstration breaks down for the polygons of Figs. 6 and 7. For them, we are forced to follow the standard procedure of decomposing S into primitive triangles each of area $1/2$, together with a simple combinatorial argument. We state the general result as a theorem.

EXTENDED PICK'S THEOREM. *Let S be a lattice polygon (simple or not). Then*

$$A(S) = v - \frac{1}{2}e_b - \chi,$$

where v is the total number of lattice points in S, e_b is the number of edges on the boundary of S, and χ is the Euler characteristic of S.

Proof. We suppose that S has been decomposed into primitive triangles (see [4] for a proof that this can be done) and let v, e, and f denote the number of vertices, edges, and faces in this decomposition. Each triangle has 3 edges and each edge is shared by 2 triangles except for those edges on the boundary of S. Thus, $3f = 2e - e_b$ and so

$$f = -e_b + 2e - 2f = 2v - e_b - 2(v - e + f) = 2v - e_b - 2\chi.$$

We conclude that

$$A(S) = \frac{1}{2}f = v - \frac{1}{2}e_b - \chi. \blacksquare$$

We have recently learned of an even more general form of Pick's theorem; it appears in reference [3].

References

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UNBOUNDED SEQUENCES OF EULER-DEDEKIND MEANS

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Euler's totient function, the number of integers not exceeding n and prime to it, is well known to be given by

$$\phi(n) = n\Pi(1 - 1/p),$$

where the product is taken over the distinct primes that divide n. Less well known is the closely related Dedekind function:

$$\psi(n) = n\Pi(1 + 1/p).$$

Define $g(n)$ to be the mean of these:

$$g(n) = (\phi(n) + \psi(n))/2.$$

It is easy to see that $g(n)$ is always an integer and that $g(n) = n$ whenever n is a prime power. So, if we iterate g, the resulting sequence may become constant:

$$g(12) = 14,\ g(14) = 15,\ g(15) = 16,\ g(16) = 16, \dots .$$

Richard Guy [1] asked if such sequences can increase indefinitely; we show that the answer is "yes."

Let $x_{n+1} = g(x_n)$ with $x_1 = 1488 = 2^4 3 \cdot 31$. Then

$$x_2 = 2^4 3 \cdot 37, \quad x_3 = 2^6 3 \cdot 11, \quad x_4 = 2^6 41, \quad x_5 = 2^5 83, \quad x_6 = 2^4 167,$$
$$x_7 = 2^3 5 \cdot 67, \quad \text{and} \quad x_8 = 2^5 3 \cdot 31 = 2x_1,$$

so that for $n \geqslant 1$, we have $x_{n+7} = 2x_n$ and the sequence $\{x_n\}$ is unbounded.

The two smallest initial values of such unbounded sequences are $x_1 = 45$ and $x_1 = 50$. Each of these gives $x_3 = 56$, $x_{28} = 1488$, so that for $n \geqslant 28$, we again have $x_{n+7} = 2x_n$.

Reference

1. Richard K. Guy, Unsolved Problems in Number Theory, Springer-Verlag, New York, 1981.

155. **MISCELLANEA**

What is the next element of the sequence that begins with F4E, S9, SE5EN, ...?

—Donald E. Knuth

See p. 595.