Steps towards the Stirling approximation for $n!$

Math 496
2/4/19

1. Fun with integration by parts. Let f be any function for which f'' exists

$$\int_{x=0}^{1} f(x)\,dx \qquad u = f(x) \qquad dv = dx$$
$$du = f'(x)dx \qquad v = x - \tfrac{1}{2} \;(\leftarrow \text{"trick me"})$$

$$= f(x)(x-\tfrac{1}{2})\Big|_{x=0}^{1} - \int_0^1 (x-\tfrac{1}{2})f'(x)dx = \frac{f(0)+f(1)}{2} - \int_0^1 (x-\tfrac{1}{2})f'(x)dx$$

$$\int_0^1 (x-\tfrac{1}{2})f'(x)dx \qquad u = f'(x) \qquad dv = x - \tfrac{1}{2}\,dx$$
$$du = f''(x)dx \qquad v = \frac{x^2-x}{2}.$$

$$= f'(x)\cdot\frac{x^2-x}{2}\Big|_0^2 - \int_0^1 \frac{x^2-x}{2} f''(x)dx = f'(0)\cdot 0 + f(0)0 - \int_0^1 \frac{x^2-x}{2} f''(x)dx$$

Putting this together $\int_0^1 f(x)dx = \frac{f(0)+f(1)}{2} - (-)\int_0^1 \frac{x^2-x}{2} f''(x)dx$

2. Special case. $f(x) = \log(x+k)$, $k \geq 1$, $f'(x) = \frac{1}{x+k}$, $f''(x) = \frac{-1}{(x+k)^2}$

So $\int_0^1 \log(x+k)dx = \frac{\log k + \log k+1}{2} + \int_0^1 \frac{1}{2}\cdot\frac{x^2-x(-1)}{(x+k)^2}dx$

Let $b_k = \int_0^1 \frac{1}{2}\frac{x^2-x}{(x+k)^2}dx$ Then $|b_k| \leq \int_0^1 \max\frac{|x^2-x|}{2}\cdot\max\frac{1}{(x+k)^2}\,dx$
$$= \frac{1}{8}\cdot\frac{1}{k^2}.$$

So $\int_k^{k+1} \log x\,dx = \frac{\log k + \log(k+1)}{2} + b_k$, where $|b_k| \leq \frac{1}{8k^2}$

Sum this from $k=1$ to $n-1$.

$$\int_1^n \log x\,dx = \frac{\log 1 + \log 2}{2} + \frac{\log 2 + \log 3}{2} + \cdots + \frac{\log n-1 + \log n}{2} + (b_1 + \cdots + b_{n-1})$$

Let $S_n = b_1 + \cdots + b_{n-1}$. We know $\sum b_k$ is convergent, so $S_n \to S$

$(n\log n - n) - (1\log 1 - 1) = \log 2 + \log 3 + \cdots + \log n-1 + \frac{1}{2}\log n + S_n$
$$= \log n! - \tfrac{1}{2}\log n + S_n$$

To sum up $\log n! = n\log n - n + 1 + \frac{1}{2}\log n - S_n$

$$\log n! = (n+\tfrac{1}{2})\log n - n + (1 - S_n)$$

3. Wallis' identity

Let $I_n = \int_0^{\pi/2} \cos^n\theta\, d\theta$. Then $I_0 \geq I_1 \geq I_2 \geq \cdots$ since $0 \leq \cos\theta \leq 1$.

$I_0 = \frac{\pi}{2}$ $\quad I_1 = \int_0^{\pi/2} \cos\theta\, d\theta = \sin\theta\Big|_0^{\pi/2} = 1.$

$\underline{n \geq 2}$ $\quad I_n = \int_0^{\pi/2} \cos^{n-1}\theta \cdot \cos\theta\, d\theta.$ $\quad u = \cos^{n-1}\theta$ $\quad dv = \cos\theta\, d\theta$

$du = -(n-1)\sin\theta \cdot \cos^{n-2}\theta\, d\theta$ $\quad v = \sin\theta$

$$= \cos^{n-1}\theta \cdot \sin\theta\Big|_0^{\pi/2} - \int_0^{\pi/2} -\sin^2\theta\,(n-1)\cos^{n-2}\theta\, d\theta$$

$$= 0 - 0 + (n-1)\int_0^{\pi/2} \sin^2\theta \cos^{n-2}\theta\, d\theta = (n-1)\int_0^{\pi/2} (1-\cos^2\theta)\cos^{n-2}\theta\, d\theta$$

So $I_n = (n-1)(I_{n-2} - I_n) \Rightarrow n I_n = n-1\, I_{n-2} \Rightarrow I_n = \frac{n-1}{n} I_{n-2}$

$I_0 = \frac{\pi}{2}$, $\; I_2 = \frac{\pi}{2}\cdot\frac{1}{2}$, $\; I_4 = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}$, $\; I_6 = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}$ $\;\ldots$

$I_1 = 1$, $\; I_3 = 1\cdot\frac{2}{3}$, $\; I_5 = 1\cdot\frac{2}{3}\cdot\frac{4}{5}$, $\; I_7 = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}$

Note:

$I_{2n} I_{2n+1} = \frac{\pi}{2}\cdot\frac{1}{2n+1}$

$I_{2n-1}\cdot I_{2n} = \frac{\pi}{2}\cdot\frac{1}{2n}$

Aid to evaluation

$$2\cdot 4 \cdots 2n = 2\cdot 1\cdot 2\cdot 2 \cdots 2\cdot n = 2^n \cdot n!$$

$$1\cdot 3\cdot 5 \cdots 2n-1 = \frac{1\cdot 2 \cdots 2n}{2 \cdots 2n} = \frac{(2n)!}{2^n n!}$$

So $I_{2n} = \dfrac{\pi}{2}\cdot\dfrac{\frac{(2n)!}{2^n n!}}{2^n n!} = \dfrac{\pi}{2}\cdot\dfrac{1}{2^{2n}}\cdot\dfrac{(2n)!}{(n!)^2}$

$$I_{2n+1} = \frac{2^n \cdot n!}{\frac{(2n+2)!}{2^{n+1}(n+1)!}} = \frac{2^{2n+1}\, n!\,(n+1)!}{(2n+2)!} = \frac{2^{2n+1}\, n!\cdot n!\,(n+1)}{(2n)!\,(2n+1)(2n+2)}$$

$$= \frac{2^{2n}(n!)^2}{(2n)!\,(2n+1)}$$

So, to give a sense of scale,

$I_{2n}^2 \le I_{2n-1} I_{2n} = \frac{\pi}{2} \cdot \frac{1}{2n} \Rightarrow I_{2n} \to 0$. Similarly, $I_{2n+1} \to 0$.

We have $\frac{I_{2n}}{I_{2n+1}} \geq 1 \geq \frac{I_{2n}}{I_{2n-1}}$

and $\frac{I_{2n}}{I_{2n+1}} = \frac{\pi}{2} \cdot \frac{1}{2^{4n}} \cdot \frac{((2n)!)^2}{(n!)^4} (2n+1)$, $\frac{I_{2n}}{I_{2n-1}} = \frac{\pi}{2} \cdot \frac{1}{2^{4n}} \cdot \frac{((2n)!)^2}{(n!)^4} 2n$

only difference

4. From 2, $n! = \frac{n^{n+\frac{1}{2}}}{e^n} e^{1-S_n} = \frac{n^n}{e^n} \sqrt{n} \cdot C_n$, where $C_n = e^{1-S_n} \to e^{1-S}$

exact formula

Thus $\frac{2n!}{(n!)^2} = \frac{(2n)^{2n}}{e^{2n}} \cdot \sqrt{2n} \cdot C_{2n} \Big/ \frac{n^{2n}}{e^{2n}} (\sqrt{n})^2 \cdot C_n^2$

Lots of cancelation $\frac{(2n)!}{(n!)^2} = 2^{2n} \cdot \frac{\sqrt{2}}{\sqrt{n}} \cdot \frac{C_{2n}}{C_n^2}$

so $\frac{1}{2^{4n}} \frac{(2n!)^2}{(n!)^4} = \frac{1}{2^{4n}} \cdot 2^{4n} \cdot \frac{2}{n} \cdot \frac{C_{2n}^2}{C_n^4} = \frac{2}{n} \cdot \frac{C_{2n}^2}{C_n^4}$

and $\frac{\pi}{2} \cdot \frac{2}{n} \cdot \frac{C_{2n}^2}{C_n^4} \cdot (2n+1) \geq 1 \geq \frac{\pi}{2} \cdot \frac{2}{n} \cdot \frac{C_{2n}^2}{C_n^4} \cdot 2n$

$2\pi \frac{C_{2n}^2}{C_n^4} \left(1 + \frac{1}{2n}\right) \geq 1 \geq 2\pi \cdot \frac{C_{2n}^2}{C_n^4}$. $\quad C = \lim C_n$

Let $n \to \infty$ $\quad \frac{2\pi C^2}{C^4} \geq 1 \geq 2\pi \frac{C^2}{C^4} \Rightarrow C = \sqrt{2\pi}$

So $\frac{n!}{\frac{n^n}{e^n} \sqrt{2\pi n}} = \frac{\frac{n^n}{e^n} \sqrt{n}\, C_n}{\frac{n^n}{e^n} \sqrt{2\pi n}} = \frac{C_n}{\sqrt{2\pi}} \to 1.$