It is not necessary to turn in solutions to ungraded problems. All book problems are from Chapter Four. The symbol (E) indicates that a problem is either an old test question or could easily have been on a test. If you want to prepare your solution using TeX, please put your name in your own handwriting. I never mind if you put equations in by hand. Write your entire assignment carefully and clearly. It is always a good idea to look at numerical versions of a proposed result. Do not collaborate on the extra credit problem!

(ungraded) Strayer – Chapter Four – 1ac, 5a, 10a
Wolfram|Alpha problem: Pick two random integers, k and ℓ, between 100 and 1000, and ask for $m = \text{Prime}[k]$ and $n = \text{Prime}[k]$ Now ask for $(m\ n)$ and $(n\ m)$, either by asking if these are quadratic residues, or by using the formulation “JacobiSymbol[m,n].”

1. Strayer – Problem 1bd.
2. Strayer – Problem 10bd.
3. This problem uses your number $N$. Determine the value of the Legendre symbol $(N\ 11)$. You should be able to do this from the definition, without using quadratic reciprocity.
4 & 5. (E) The quadratfrei (= “squarefree” in German) function $Q(n)$ is defined as follows:

\[ Q(1) = 1; \quad n = \prod_{j=1}^{r} p_j^{a_j} \implies Q(n) = \prod_{j=1}^{r} p_j. \]

That is, $Q(n)$ is equal to the product of the primes which divide $n$. (Above, we assume that the primes in the expression for $n$ are distinct and $a_j \geq 1$.)

a. Prove that $Q$ is multiplicative.

b. For any prime $p$, write down $Q_p(T)$, as defined in the class notes.

c. Determine $(Q \ast 1)(n) = \sum_{d|n} Q(d)$ as a straightforward “closed” expression based on the prime factorization of $n$. For example

\[ \sum_{d|12} Q(d) = Q(1) + Q(2) + Q(3) + Q(4) + Q(6) + Q(12) = 1 + 2 + 3 + 2 + 6 + 6 = 20. \]

Hint: first, look at $n = p^a$.

d. Prove that

\[ (Q \ast \mu)(n) = \prod_{j=1}^{r} (p_j - 1) \]

Please turn over!
6 & 7. (E) Let $\beta(n)$ be the “Generous Möbius” function defined by:

$$\beta(1) = 1; \quad n = \prod_{j=1}^{r} p_j^{a_j} \text{ and } a_j \in \{1, 2\} \implies \beta(n) = (-1)^r;$$

$$p^3 \mid n \quad \text{for any prime } p \implies \beta(n) = 0.$$ 

a. Prove that $\beta$ is multiplicative.

b. For any prime $p$, write down $\beta_p(T)$, as defined in the class notes.

c. Determine $(\beta \star 1)(n) = \sum_{d \mid n} \beta(d)$ as a straightforward “closed” expression based on the prime factorization of $n$. For example

$$\sum_{d \mid 12} \beta(d) = \beta(1) + \beta(2) + \beta(3) + \beta(4) + \beta(6) + \beta(12) = 1 + (-1) + (-1) + (-1) + (1) + 1 = 0.$$ 

Hint: first find $(\beta \star 1)(p^a)$ . Expect different answers for $a = 1$ and $a \geq 2$.

8. (Extra credit, 1/2 point) Using the notation of the last problem, suppose $z(n)$ is an multiplicative function defined by $\beta \star z = \delta$; that is, $(\beta \star z)(1) = 1$ and $(\beta \star z)(n) = 0$ if $n > 1$. Express $z(n)$ for $n = \prod_{j=1}^{r} p_j^{a_j}$ in terms of the Fibonacci sequence: $F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. No hints, other than the Fibonacci clue.