1. Only noted neg. error, on p.3 $2 \cdot 3 \cdot 5 = 60$, not so

Before we do convolution inverses, one point to be made:
We had $F = f \circ \mu$; $F(n) = \sum f(k) \mu(n/k)$. Mobius Inversion
simply says that we wrote

$$F \mu = (f \circ \mu) \circ \mu = f \circ (I \circ \mu) = f \circ \theta = f.$$ 

Next is,

$$f(n) = \sum_{d \mid n} f(d) \mu(n/d) \iff f(n) = \sum_{d \mid n} F(d) \mu(n/d)$$

In content, this isn’t so surprising!

$M = \{ f : f \text{ is multiplicative } \text{ and } f(1) = 1 \}$ (i.e., $f \neq 0$).

**Theorem**

If $f \in M$, then there exists $g \in M$ such that $f \circ g = \delta$

**Proof.**

Suppose we define $g$ so that for any prime $p$ and $a \geq 1$,

$$(p^a) \cdot (f \circ g)(p^a) = 0.$$ 

This means knowing $g(p^k)$ for primes $p$ and $k > 1$, $g(1) = 1$.

Then define $g$ as a multiplicative function by

$$g(n) = g(p_1^{a_1} \cdot \ldots \cdot p_r^{a_r}) := g(p_1^{a_1}) \cdot \ldots \cdot g(p_r^{a_r})$$

to get what we need.

What do we need? $1 = (f \circ g)(1) = f(1)g(1) \iff g(1) = 1$

$0 = f \circ g(p) = f(1)g(p) + f(p)g(1) \Rightarrow g(p) = -f(p)$, so we take this as our definition of $g(p)$.
Now, we assume that $g(1), g(p), \ldots, g(p^{m-1})$ have been defined already. Define

$$g(p^n) = -f(p)g(p^{n-1}) - f(p^2)g(p^{n-2}) - \ldots - f(p^n)g(1)$$

Recall, $f$ is known and $g(1), g(p), \ldots, g(p^{m-1})$ are already known, so $g(p^n)$ is determined, ad

$$(f+g)(p^m) = f(p^m)g(p^m) + f(p)g(p^{m-1}) + \ldots + f(p^n)g(1) = 0.$$

Example. Let $f = 1$, so $f(p^n) = 1$ for all $p^n$.

1. $g(1) = 1$
2. $g(p) = f(p) = -1$
3. $g(p^2) = -f(p)g(p) = -f(p^2)g(1) = (-1) \cdot 1 = -1$
4. $g(p^3) = -f(p)g(p) - f(p^2)g(p) - f(p^3)g(1) = g(p^3) = 0$;

It is not hard to see that $g = M$.

Corollary

$(M, +)$ is an abelian group.

Proof.

If $f, g \in M$, then $f + g \in M$ (closure), $(f + g) + h = f + (g + h)$ (associativity), $f + 0 = f$ (identity) and $f + (-f) = 0$ (inverse)

$f + g = g + f$ (commutative, so abelian)

Next, a short try to generating function.
Suppose \( f \) is multiplicative and \( p \) is prime. We have

\[
f_p(t) = 1 + f(p)t + f(p^2)t^2 + f(p^3)t^3 + \ldots.
\]

**DON'T WORRY ABOUT CONVERGENCE!** (This is called a "formal" power series, as is studied in grad school.)

**Why Do This?**

**THEOREM** \([\log]_p(T) = f_p(T) \cdot g_p(T)\)

**Proof sketch**

\[
f_p(t) = 1 + f(p)t + f(p^2)t^2 + f(p^3)t^3 + \ldots
\]

\[
g_p(t) = 1 + g(p)t + g(p^2)t^2 + g(p^3)t^3 + \ldots
\]

Now multiply and collect terms by powers of \( t \).

\[
f_p(t)g_p(t) = 1 + [f(p) \cdot 1 + g(p)]t + [f(p^2) \cdot 1 + f(p)g(p) + 1 \cdot g(p^2)]t^2 + [f(p^3) \cdot 1 + f(p^2)g(p) + f(p)g(p^2) + 1 \cdot g(p^3)]t^3 + \ldots
\]

The coefficient of \( t^k \) is \( f(p) \cdot 1 + f(p^k)g(p^k) + \ldots
\]

\[
= f(p) + \sum_{k=1}^{\infty} f(p^k)g(p^k) = (f + g)(p^k)
\]

More harems

\[
el_p(t) = 1 + t + t^2 + t^3 + \ldots = \frac{1}{1-t}
\]

\[
ep(t) = 1 + \mu(p)t + \mu(p^2)t^2 + \mu(p^3)t^3 + \ldots = 1 + \sum_{k=1}^{\infty} \mu(p^k)t^k = \frac{1}{1-t}
\]

Notice that \( \frac{1}{1-t} \cdot 1 - T = 1 \), as it should
It can get more complicated. Recall that $N(a) = n$, so

\[ N_p(t) = 1 + N(p)t + N(p^2)t^2 + N(p^3)t^3 + \cdots = \frac{1}{1 - pt} \]

\[ \Phi = N \ast \mu, \text{ so} \]

\[ \Phi_p(t) = N_p(t) \cdot \mu_p(t) = \frac{1 - t}{1 - pt} - \frac{1}{p} + \frac{1 - \frac{1}{p}}{1 - pt} \text{ (Think techniques of integrals, say partial fraction)} \]

Hence for $k \geq 1$

\[ \Phi(p^k) = \left( 1 - \frac{1}{p} \right) - p^k = p^k - p^{k-1} \text{ (sounds right)} \]

\[ \nabla = 1 + N, \text{ so} \]

\[ \nabla_p(t) = \Phi_p(t) \cdot N_p(t) = \frac{1}{1 - t} \cdot \frac{1}{1 - pt} \text{. Again, techniques of integrals, say partial fraction} \]

\[ \frac{1}{1 - t} \cdot \frac{1}{1 - pt} = \frac{p^{1-p-1}}{1 - pt} - \frac{1^{1-p-1}}{1 - t} \text{, so the coefficient at } t^k \text{ gives} \]

\[ \nabla_p(p^k) = \frac{p^k - p^{k-1}}{p - 1} \]

A different tack:

Lemma: If $f$ and $g$ are multiplicative functions, so is $fg$

Proof: Suppose $gcd(m, n) = 1$. Then

\[ (fg)(mn) = f(mn)g(mn) = f(m)f(n)g(m)g(n) = f(m)f(n)g(m)g(n) = fg(m)f(g(n)) \]

def. of $f$ and $g$, mutl. assoc., def.

Fun fact: If $f \in M$, then $f + J = f$ but $fJ = J$!

Fun fact 2: $(Nf) * (Ng) = N(f * g)$.

\[ Pf \]

\[ (Nf) + (Ng)(n) = \sum_{d|n} (Nf)(d)(Ng)(\frac{n}{d}) = \sum_{d|n} (df/\alpha)(d)g(\frac{n}{d}) \]

\[ = n \sum \frac{f(d)g(\frac{n}{d})}{d} = n(f * g)(n) = (Nf + Ng)(n) \]
More interesting multiplicative functions!

Since \( \mu \) is multiplicative, \( \mu \cdot \mu = \mu^2 \) is multiplicative and since \( \mu^2(n) = 1 \) if \( n = \prod_i p_i^{a_i} \), we have \( \mu^2(n) = 1 \Leftrightarrow \exists i \text{ such that } p_i^2 \mid n \). Therefore:

\[
\mu^2(n) = \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right) \quad \text{so} \quad (\mu^2(n) = 1) \Leftrightarrow P^2 \mid n. 
\]

For example:

\[
\mu_p(T) = 1 + \mu_p(T) T + \mu_p(T^2) T^2 + \cdots = 1 + T.
\]

\[
\mu_p(T) \cdot 1_p(T) = 1 + T^2 + 2T^4 + \cdots = 1 + 2T + 2T^2 + 2T^3 + \cdots 
\]

so

\[
(\mu_p(T) \cdot 1_p(T))^k \sum_{k=0}^{+\infty} T^k = 1 + \sum_{k=0}^{+\infty} T^k
\]

Hence

\[
(\mu_p(T) \cdot 1_p(T))^k \sum_{k=0}^{+\infty} T^k = 1 + \sum_{k=0}^{+\infty} T^k 
\]

This function is clearly multiplicative.

If \( n = \prod_{i=1}^{r} p_i^{a_i} \), then \( \mu(n) = \prod_{i=1}^{r} (-1)^{a_i} \).

Another natural multiplicative function, call it \( h \), has

\[
h(\prod_{i=1}^{r} p_i^{a_i}) = \prod_{i=1}^{r} p_i^{a_i}, \text{ so } h \text{ is just the product of the primes that divide } n. \text{ Since } h(p^a) = p, \text{ we have}
\]

\[
h_p(T) = 1 + pT + p^2T^2 + \cdots = \frac{1}{1 - pT} = \frac{1 + pT}{1 - T}
\]

In fact, let \( \alpha(p) \) be any function defined on the primes.

Define

\[
f(\prod_{i=1}^{r} p_i^{a_i}) = \alpha(p_1) \cdots \alpha(p_r) \text{ will be multiplicative.}
\]

Hence \( \alpha \) \& \( \alpha \otimes \) multiplicative functions.