Math 448  Homework 6  Due Friday, March 11, 2005

Please note due date. Any material on this homework that was not covered on a previous homework will not be on the first exam.

1. (ungraded) §2.4 – Problems 3, 5.
2. (ungraded) §2.4 – Problems 9, 15.
3. (ungraded) §2.5 – Problems 1, 3.

4. §2.4 – Problems 2, 4.
5. §2.4 – Problems 10, 12.
6. §2.5 – Problems 2, 6.
7. (E) Evaluate the following integrals (contours taken counterclockwise):

\[
\frac{1}{2\pi i} \int_{|z|=2} \left( z + \frac{1}{z} \right)^3 dz; \quad \frac{1}{2\pi i} \int_{|z|=2} \frac{dz}{z^2 - 3z}.
\]

8. (E) We have seen that, for \(|z| < 1\),

\[
\frac{z}{(1 - z)^2} = z + 2z^2 + 3z^3 + \cdots = \sum_{n=1}^{\infty} nz^n.
\]

Using this fact (and what we know about differentiating power series) to find polynomials \(g(z)\) and \(h(z)\) so that for \(|z| < 1\),

\[
\sum_{n=1}^{\infty} n^2 z^n = \frac{g(z)}{(1 - z)^3}; \quad \sum_{n=1}^{\infty} n^3 z^n = \frac{h(z)}{(1 - z)^4}.
\]

9. (E) Using your results in the last problem, find the power series for \(f(z) = \frac{1 + 2z}{(1 - z)^2}\) at \(z = 0\). (Hint: partial fractions.)

10. (E) Evaluate the following integrals, where \(C\) denoted the contour \(|z| = 2\) taken in the usual counterclockwise way.

\[
\frac{1}{2\pi i} \int_{C} \cos \frac{z}{z} \, dz, \quad \frac{1}{2\pi i} \int_{C} \sin \frac{z}{z} \, dz, \quad \frac{1}{2\pi i} \int_{C} \frac{e^{3z}}{z^4} \, dz.
\]

11. (E) Suppose \(C\) is a simple, piecewise smooth (not necessarily closed) contour. Prove that \(\int_{C} z \, dz = 0\) implies that \(\int_{C} z^3 \, dz = 0\).

12. (E) Find a simple, piecewise smooth contour \(C\) with the property that \(\int_{C} z \, dz = 1\) and \(\int_{C} z^3 \, dz = 0\).

13. Find all entire functions \(f(z)\) with the property that \(f(0) = 7\) and

\[
\text{Re}(f(z)) + 3 \text{ Im}(f(z)) \leq 11.
\]