
Solutions are given, and computational details are on a separate Mathematica handout

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1. This homework is due on Oct. 2, 2017 or 2 Oct. 2017 if you are European. Find, by any correct method, $s(10022017)$ and $s(2102017)$.

$\longrightarrow$ I get $s(10022017) = 7379$, $s(2102017) = 573$ via computer. Observe also that, e.g.,

$$2102107 \sim [1, 8, 1, 2, 2, 7, 1]_2 = 2^{22} - 2^{21} + 2^{13} - 2^{12} + 2^{10} - 2^8 + 2^1 - 2^0.$$

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2. Determine n so that $s(n) = 121$ and $s(n+1) = 100$. You can use the unproved formulas in the first section, but you should somehow make a persuasive argument that you have the correct answer.

$\longrightarrow$ I get $n = 15905$ from using the algorithm; see the handout. There will be a generalization on the next homework!

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3. We have seen that $s(2^r + 1) = r + 1$ for all integers $r \geq 0$. Find formulas for $s(2^r + k)$, $k = 2, 3, 4, 5, 6, 7, 8, 9, 10$, taking a bit of care about the range of r. Feel free to speculate about a general solution. Note for example that $s(2^r + 2) = s(2^{r-1} + 1)$, $s(2^r + 3) = s(2^{r-1} + 1) + s(2^{r-1} + 2)$, etc.

$\longrightarrow$ See the handout for data. We have

$$s(2^r + 1) = r + 1, \quad r \geq 0$$

so, immediately,

$$s(2^r + 2) = s(2^{r-1} + 1) = r - 1 + 1 = r, \quad r - 1 \geq 0 \iff r \geq 1$$

Note that you need the change in range, because $s(2^0 + 2) = s(3) = 2 \neq 0$! In the same way, $s(2^r + 4) = r - 1, r \geq 2$ and $s(2^r + 8) = r - 2, r \geq 3$. We also have

$$s(2^r + 3) = s(2^{r-1} + 1) + s(2^{r-1} + 2) = r + (r - 1) = 2r - 1 \iff r \geq 2$$

and, similarly,

$$s(2^r + 5) = s(2^{r-1} + 2) + s(2^{r-1} + 3) = r - 1 + 2(r - 1) - 1 = 3r - 4 \iff r \geq 3$$

The others follow. To repeat, for r sufficiently large (looks like $k \leq 2^r$), we have $s(2^r + k) = r+1, r, 2r-1, r-1, 3r-4, 2r-3, 3r-5, r-2, 4r-9, 3r-7$. Darn it, look at the coefficients of r. I think I know a sequence that goes $1, 1, 2, 1, 3, 2, 3, 1, 4, 3, \ldots$.

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4. Suppose $n = 2^r * (2m + 1)$ for integers $r, m \geq 0$. Prove that

$$s(n-1) + s(n+1) = (2r + 1)s(n).$$

$\longrightarrow$ No computer help needed. If $r = 0$, then $n = 2m+1$ and $s(2m)+s(2m+2) = s(2m+1)$, By induction, if $n = 2n'$, then

$$\frac{s(n-1)+s(n+1)}{s(n)} = \frac{s(2n'-1)+s(2n'+1)}{s(2n')} = \frac{s(n'-1)+s(n')+s(n')+s(n'+1)}{s(n')}$$
$$= \frac{s(n'-1)+s(n'+1)}{s(n')} + 2.$$

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5. Define a sequence $(b(n))$ by

$$b(0) = 0, \quad b(1) = 1; \qquad b(2n) = -b(n), \quad b(2n+1) = b(n) + b(n+1) \text{ for } n \geq 1.$$

a. Calculate $(b(n))$ for $2 \leq n \leq 8$ and formulate a conjecture about a (simple) closed formula for $ba(n)$). Then prove it by induction.

$\longrightarrow$ The sequence seems to obey the property that $b(3n) = 0$, $b(3n+1) = 1$ and $b(3n+2) = -1$. This is easily proved by induction upon looking at $b(n)$ for $n \mod 6$. (Proof omitted.)

b. By writing

$$B(x) := \sum_{n=0}^{\infty} b(n)x^n = \sum_{n=0}^{\infty} b(2n)x^{2n} + \sum_{n=0}^{\infty} b(2n+1)x^{2n+1}, \qquad B(x) = xC(x),$$

find a formula relating $C(x)$ with $C(x^2)$. Use this formula to get a closed form for $B(x)$, which is then an alternate proof for your answer in a. Use some algebraic identities!

$\longrightarrow$ We have

$$B(x) = \sum_{n=0}^{\infty} b(2n)x^{2n} + \sum_{n=0}^{\infty} b(2n+1)x^{2n+1}$$
$$= -\sum_{n=0}^{\infty} b(n)x^{2n} + \sum_{n=0}^{\infty} b(n)x^{2n+1} + \sum_{n=0}^{\infty} b(n+1)x^{2n+1}$$
$$= -B(x) + xB(x^2) + x^{-1}B(x^2) \implies B(x) = (-1+x+x^{-1})B(x^2)$$
$$\implies xB(x) = (-1+x+x^{-1})x^2B(x^2) \implies C(x) = (1-x+x^2)C(x^2)$$
$$\implies C(x) = \left(\prod_{k=0}^{N-1}\left(1-x^{2^k}+x^{2\cdot 2^k}\right)\right)\cdot C(x^{2^N}) \implies C(x) = \prod_{k=0}^{\infty}\left(1-x^{2^k}+x^{2\cdot 2^k}\right)$$
$$\implies B(x) = x\prod_{k=0}^{\infty}\left(1-x^{2^k}+x^{2\cdot 2^k}\right)$$

Since $\frac{1+u^3}{1+u} = 1-u+u^2$, taking $u = x^{2^k}$ yields

$$B(x) = x\prod_{k=0}^{\infty}\frac{1+x^{3\cdot 2^k}}{1+x^{2^k}} = x\frac{\prod_{k=0}^{\infty} 1+x^{3\cdot 2^k}}{\prod_{k=0}^{\infty} 1+x^{2^k}}$$

$$= x\cdot\frac{\frac{1}{1-x^3}}{\frac{1}{1-x}} = x\frac{1-x}{1-x^3} = \frac{x}{1+x+x^2} = \frac{x-x^2}{1-x^3} = (x-x^2)(1+x^3+x^6+\ldots).$$

6. Define a sequence $(v_\lambda(n))$ by

$$v_\lambda(0) = 0, \quad v_\lambda(1) = 1, \qquad v_\lambda(n) = \lambda v_\lambda(n-1) + v_\lambda(n-2), \; n \geq 2$$

where $\lambda \in \mathbb{C}$ is regarded as a parameter. (Hints: binomial coefficients will be involved; $v_1(n) = F_n$; this function is built-in to Mathematica under the name Fibonacci[n, λ].)

a. Show that for $n \geq 1$, $v_\lambda(n)$ is a polynomial of degree $n-1$ in λ, and find and prove an explicit formula for its coefficients.

$\longrightarrow$ The first few $v_\lambda's$ are: $0, 1, \lambda, \lambda^2+1, \lambda^3+2\lambda$. It is easy to see by induction that these are polynomials of degree $n-1$ and a certain tilted Pascal triangle becomes evident if you write them out (see printout). Using the fact that binomial coefficients zero out if the parameters are in the wrong range, we seem to have

$$v_\lambda(n) = \sum_{k\geq 0} \binom{n-1-k}{k} \lambda^{n-1-2k}.$$

This is true for small n, and if we assume it true by induction, then

$$\lambda v_\lambda(n-1) + v_\lambda(n-2) = \lambda \sum_{j\geq 0} \binom{n-2-j}{j} \lambda^{n-2-2j} + \sum_{\ell \geq 0} \binom{n-3-\ell}{\ell} \lambda^{n-3-2\ell}$$

$$= \sum_{k\geq 0} \binom{n-2-j}{j} \lambda^{n-1-j} + \sum_{k\geq 0} \binom{n-3-\ell}{\ell} \lambda^{n-3-2\ell}$$

All powers of λ which appear above have the same parity as $n-1$, and the coefficient of λ^{n-1-2k} occurs above when $j = k, \ell = k-1$, and so is

$$\binom{n-2-k}{k} + \binom{n-3-(k-1)}{k-1} = \binom{n-2-k}{k} + \binom{n-2-k}{k-1} = \binom{n-1-k}{k},$$

as desired.

b. Prove by any correct method that

$$v_\lambda^2(n) = v_\lambda(n-1)v_\lambda(n+1) + (-1)^n \qquad \text{or, ahem, } (-1)^{n+1}$$

$\longrightarrow$ You could try to prove this by the formula above, or by the closed formula (see handout). One fast way is to observe that for $n = 2$, the assertion is that $\lambda^2 = 1 * (\lambda^2+1) + (-1)^1$, which I believe. Now define

$$W_\lambda(n) = v_\lambda^2(n) - v_\lambda(n-1)v_\lambda(n+1)$$

We want $W_\lambda(n) = (-1)^{n+1}$; since $W_\lambda(2) = -1$, it suffices to prove that

$$W_\lambda(n) + W_\lambda(n+1) = 0.$$

But the left hand side is

$$v_\lambda^2(n) - v_\lambda(n-1)v_\lambda(n+1) + v_\lambda^2(n+1) - v_\lambda(n)v_\lambda(n+2)$$

$$= v_\lambda^2(n+1) - v_\lambda(n-1)v_\lambda(n+1) - (v_\lambda(n)v_\lambda(n+2) - v_\lambda^2(n))$$

$$= v_\lambda(n+1)\big(v_\lambda(n+1) - v_\lambda(n-1)\big) - v_\lambda(n)\big(v_\lambda(n+2) - v_\lambda(n)\big)$$

$$= v_\lambda(n+1)(\lambda v_\lambda(n)) - v_\lambda(n)(\lambda v_\lambda(n+1)) = 0!$$