Math 428 **Homework 1** **Due Monday, October 2, 2017**

It's ok to work together, but don't copy without comprehension. It will be helpful to have the assistance of a program like Mathematica. I've already given a handout with the recursive definition of the Stern sequence from there. If you use a different program, please also tell me how you define the sequence!

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1. This homework is due on Oct. 2, 2017 or 2 Oct. 2017 if you are European. Find, by any correct method, $s(10022017)$ and $s(2102017)$.

2. Determine n so that $s(n) = 121$ and $s(n+1) = 100$. You can use the unproved formulas in the first section, but you should somehow make a persuasive argument that you have the correct answer.

3. We have seen that $s(2^r + 1) = r + 1$ for all integers $r \geq 0$. Find formulas for $s(2^r + k)$, $k = 2, 3, 4, 5, 6, 7, 8, 9, 10$, taking a bit of care about the range of r. Feel free to speculate about a general solution. Note for example that $s(2^r + 2) = s(2^{r-1} + 1)$, $s(2^r + 3) = s(2^{r-1} + 1) + s(2^{r-1} + 2)$, etc.

4. Suppose $n = 2^r * (2m + 1)$ for integers $r, m \geq 0$. Prove that

$$s(n-1) + s(n+1) = (2r+1)s(n).$$

5. Define a sequence $(b(n))$ by

$$b(0) = 0, \quad b(1) = 1; \qquad b(2n) = -b(n), \quad b(2n+1) = b(n) + b(n+1) \text{ for } n \geq 1.$$

a. Calculate $(b(n))$ for $2 \leq n \leq 8$ and formulate a conjecture about a (simple) closed formula for $ba(n))$. Then prove it by induction.
b. By writing

$$B(x) := \sum_{n=0}^{\infty} b(n)x^n = \sum_{n=0}^{\infty} b(2n)x^{2n} + \sum_{n=0}^{\infty} b(2n+1)x^{2n+1}, \qquad B(x) = xC(x),$$

find a formula relating $C(x)$ with $C(x^2)$. Use this formula to get a closed form for $B(x)$, which is then an alternate proof for your answer in a. Use some algebraic identities!

6. Define a sequence $(v_\lambda(n))$ by

$$v_\lambda(0) = 0, \quad v_\lambda(1) = 1, \qquad v_\lambda(n) = \lambda v_\lambda(n-1) + v_\lambda(n-2), \ n \geq 2$$

where $\lambda \in \mathbb{C}$ is regarded as a parameter. (Hints: binomial coefficients will be involved; $v_1(n) = F_n$; this function is built-in to Mathematica under the name Fibonacci$[n, \lambda]$.)
a. Show that for $n \geq 1$, $v_\lambda(n)$ is a polynomial of degree $n-1$ in λ, and find and prove an explicit formula for its coefficients.
b. Prove by any correct method that

$$v_\lambda^2(n) = v_\lambda(n-1)v_\lambda(n+1) + (-1)^n.$$