Math 428, Ninth worksheet, 10/18/17

1a. Suppose we have a generating function

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

(Easy) Prove that

$$\frac{G(x) + G(-x)}{2} = \sum_{n=0}^{\infty} a_{2n} x^{2n}, \qquad \frac{G(x) - G(-x)}{2} = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}.$$

1b. Let (F_n) be the usual Fibonacci sequence ($F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.) We have already seen that

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}.$$

Use 1a. to give closed forms (that is, rational functions) for

$$\sum_{n=0}^{\infty} F_{2n} x^{2n}, \qquad \sum_{n=0}^{\infty} F_{2n+1} x^{2n+1}$$

(Computational hint: $(1 - x - x^2)(1 + x - x^2) = 1 - 3x^2 + x^4$; it's a difference of two squares.)

2a. (Building on yesterday.) Consider a sequence defined by $a_0 = 9$ and $a_n = 4a_{n-1} + 1$, $n \geq 1$. You showed yesterday that $b_n := s(a_n)$ satisfies the second-order equation $b_n = 3b_{n-1} - b_{n-2}$. A mysterious Mathematica handout gave the data $b_0 = s(9) = 4, b_1 = s(37) = 11$, and strongly suggested that

$$b_n = F_{2n+2} + F_{2n+4}.$$

Your tasks: verify this formula for $n = 0, 1$ (easy), then verify that $c_n = F_{2n+2} + F_{2n+4}$ satisfies the same recurrence $c_n = 3c_{n-1} - c_{n-2}$. Equality for all n then follows immediately by induction.

2b. Find "nice" constants α, β so that

$$b_n = c_n = \alpha F_{2n} + \beta F_{2n+1}.$$

There are any number of ways to do this.

3. We have seen that for the growth function of the Stern sequence,

$$
\begin{aligned}
F\left(\frac{1}{3}\right) &= \frac{s(2^3+1)}{2\cdot 3^2} + \frac{s(2^5+2^2+1)}{2\cdot 3^4} + \frac{s(2^7+2^4+2^2+1)}{2\cdot 3^6} + \dots \\
&= \frac{b_0}{2\cdot 3^2} + \frac{b_1}{2\cdot 3^4} + \frac{b_2}{2\cdot 3^6} + \dots \\
&= \sum_{n=0}^{\infty} \frac{b_n}{2\cdot 3^{2n+2}}.
\end{aligned}
$$

Use all parts of this exercise to convince yourself that $F\left(\frac{1}{3}\right) = \frac{7}{22}$.