Math 428, Eighth worksheet, 10/16/17

1a. We might have done this already. Let

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

and define the Fibonacci sequence as usual by $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that for integers $n \geq 1$,

$$M_1^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

(I want an induction proof, don't worry about diagonalization, etc.)

1b. Now let

$$M_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Compute M_2^2, M_2^3 and guess and prove an explicit formula for M_2^n. Hint: if you do this right, then you can use part a.

2a. Define a sequence (a_n) by $a_0 = r$ and $a_n = 4a_{n-1} + 1$. Prove that

$$\begin{pmatrix} s(a_n) \\ s(a_n + 1) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} s(a_{n-1}) \\ s(a_{n-1} + 1) \end{pmatrix}.$$

2b. Given that $a_0 = r$ is unspecified, describe the binary expansion of a_n in terms of that of r. Also, use this formulas from 1b and 2a to derive a single (second-order) recurrence satisfied by $s(a_n)$, and use this to show that there exists a constant $\alpha(r)$ such that

$$s(a_n) \approx \alpha(r) \cdot \phi^{2n}, \qquad \phi = \frac{1 + \sqrt{5}}{2}.$$