Ungraded problems will be discussed in class.

1. If \((E,d)\) were \(R^2\) with the usual distance, this would be a half-space determined by a hyperplane. (\(\star\))

It isn't.

So \(S = \{ x \in E : d(p,x) < d(q,x) \}\)

Pick \(y \in S\). We want to find \(r\) so that \(d(y,r) \leq s\) for all \(z \in B(y,r)\), then \(z \in S\).

Let \(d(p,y) = A\)

\[ d(q,y) = A + \frac{1}{R} (R > 0) \]

I claim \(d(y,z) = \frac{A}{2}\) will work.

Suppose \(z \in B(y,\frac{A}{2})\). Then

\[ d(y,z) < \frac{A}{2} \]

What about \(d(p,z), d(q,z)\)?

\[ d(p,z) \leq d(p,y) + d(y,z) < A + \frac{A}{2} \]

\[ d(q,z) \leq d(q,y) - d(y,z) > A - \frac{A}{2} = A + \frac{A}{2} \]

So \(d(q,z) > d(p,z)\) and we are done.

2. That is, given \(\varepsilon > 0\), find \(N\) so that \(N > 5\) \(\Rightarrow \sqrt{9 - \frac{1}{n} - \varepsilon} \leq 3\)

Observe that \(3 > \sqrt{9 - \frac{1}{n}} \geq \sqrt{8} \)

so if \(3 - \frac{1}{n} > \varepsilon \)

Then \(N = 1\) will do

\[ 3 - \sqrt{9 - \frac{1}{n}} < \varepsilon \]

\[ 3 - \sqrt{9 - \frac{1}{n}} < \sqrt{9 - \frac{1}{n^2}} \]

\[ 9 - 6\varepsilon + 3\varepsilon^2 < 9 - \frac{1}{n^2} \]

\[ \varepsilon^2 - 3\varepsilon^2 = 3(\varepsilon - 3\varepsilon)^2 \]

So if \(0 \leq \varepsilon \leq 3\), then \(\frac{1}{n} \leq \varepsilon < 3\)

If \(\frac{1}{n^2} < \varepsilon < 3\) and we have

\[ \frac{1}{n^2} < \varepsilon \leq (\varepsilon - 3\varepsilon)^2 \]

\[ \frac{1}{n^2} < \varepsilon^2 \]

\[ \frac{1}{n^2} < \varepsilon^2 \]

So \(N = \frac{\varepsilon^2}{2}\) will work.

Notes: There are a lot of variables. One could also solve the quadratic \(\varepsilon^2 - 6\varepsilon + \frac{1}{n^2} < 0\)

Every correctly proved \(N\) is \(\Delta^2\)!

3. A lot like 1.

Schematic

\[ \text{Let } G = B(P_1, \gamma) \text{ where } \gamma \text{ is chosen later.} \]

Clearly, \(P_1 \not\in O_2\) are open sets, \(P_1 \in G_1\) and \(P_2 \in G_2\).

Since they are disjoint:

\[ x \in O_1 \land O_2 \Rightarrow d(P_1,x) < r, d(P_2,x) > \gamma \]

Let \(r = \frac{d(P_1,P_2)}{2}\). Then by \(\Delta^2\)

Maybe unrealistic.

\[ d(P_1,P_2) \leq d(P_1,x) + d(x,P_2) < 2r = d(P_1,P_2) \]
This is a contradiction. Thus the bails Q and Q are desjoint. If you take any \( r \leq \frac{\varepsilon}{2} \), this will work too.

4. The idea is to make sure that \( d(y_m, y_n) \) is small for large \( (m, n) \), all the way to the time to trouble

\[ d(y_m, y_n) = d(y_m, x_m) + d(x_m, x_n) + d(x_n, y_n) \]

and use the triangle inequality

Given \( \varepsilon > 0 \), there exists \( N_1 \)

so that \( m, n > N_1 \Rightarrow d(x_m, x_n) < \frac{\varepsilon}{3} \) 

because \( (x_n) \) is Cauchy.

There exists \( N_2 \) so that \( r > N_2 \) 

\[ d(x_m, y_n) < \frac{\varepsilon}{3} \], because

\[ d(x_m, y_n) \Rightarrow 0 \].

Thus, if \( N > \max(N_1, N_2) \)

and \( m, n > N \),

\[ d(y_m, y_n) \leq d(y_m, x_m) + d(x_m, x_n) + d(x_n, y_n) \]

\[ < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \]

and we are done.

5. Lots of ways to derive theo

\[ f(x) = 4 + \frac{5}{x} \]

\[ a_0 = 4 \]

\[ a_n = f(a_{n-1}) \]

so \( a_1 = 5 + \frac{5}{2} \), \( a_2 = 5 - \frac{5}{4} \), \( a_3 = 5 + \frac{1}{10} \)

e tc.

Using the suggestions,

\[ a_n = 4 + \frac{\varepsilon}{a_{n-1}} = \frac{4a_{n-1} + 5}{a_{n-1}} \]

\[ b_n = 5 - a_n = 5 - \frac{4a_{n-1} + 5}{a_{n-1}} = \frac{a_{n-1} - 5}{a_{n-1}} \]

\[ b_n = -\frac{b_n-1}{5-b_n} \]

\[ C_n = \frac{1}{b_n} \]

\[ C_n = \frac{\frac{1}{a_{n-1}}}{5 - \frac{1}{a_{n-1}}} = \frac{1}{5C_{n-1}} \]

So \( C_n = 1 - 5C_{n-1} \)

\( C_0 = 1, C_1 = 1 - 5, C_2 = 1 - 5(1 - 5) = 1 - 5 + 5^2 \)

An easy induction says \( \frac{A}{n} \)

\[ C_n = \sum_{k=0}^n (-5)^k \]

\[ = (-5)^n - 1 \]

\[ = \frac{1 + (-5)^{n+1}}{6} \]

\[ = \frac{1 + (-5)^{n+1}}{6} \]

\[ a_n - 5 = \frac{6}{5^{n+1}(-1)^n} \]

If you want to use \( 5N \)

\[ 6 < \varepsilon \Rightarrow C - 10(\log 5) < \log 6 \]

\[ \frac{6 - (5 \varepsilon/6) E}{\log 5} \]

I'll accept a variety of solutions.