Graded

$52 - 12 \ x, y \subset \mathbb{R} \quad x \in X, y \in Y \Rightarrow x < y$
and $X \cup Y = \mathbb{R}$ some empty set.

Pick $u \in Y \setminus \emptyset$.
Then $u$ is an upper bound for $X$.
Hence $X$ has a least upper bound.

Let $a = \sup X$.
Suppose $a < b$. Then $a$ is not a least upper bound for $X$. Since $a$ is an upper bound, $b \in X$, subset $Y$. Suppose $c < a$, write $c = a - 2t$, $t > 0$. Suppose $c \in X$. Then $c < b$.

But $a$ is a least upper bound, so $a$ is not an upper bound for $X$.
Thus there exists $x \in X$ such that $x < d$. Since $y < x$, $d < a - 2t = c < a - t < a - \varepsilon$

so $a > c$, which is impossible.

Thus $(-\infty, a) \subseteq X$ for some $a > 0$.

$1$. Suppose $x \in T$.
Then $x = a + b$ for

Some $S \subseteq T$. Such $S$ is an upper bound for $S$, $S \subseteq T$. Hence

Thus $x$ is an upper bound.

Now suppose $t$ is an upper bound.

So $x \in T \Rightarrow x \leq T$ for all $x \in T$.

So, for all $s \in S$,

Thus $a + b - t < a + s - t - b$

$2a$. $X_0 = 12 < 16$.
If $X_n < 16$, then $X_{n+1} = 12 + \sqrt{X_n}$

Thus $X_n < 16$ for all $n$.

$2b$: Consider $X_{n+1} - X_n$

$= 12 + \sqrt{X_n} - X_n$.

Several approaches:

(1) $t < 16 \Rightarrow 12 + \sqrt{t} - t > 0$

This works.

(2) Let $y_n = \sqrt{X_n}$. Then $y_n < 4$, $y_n > 0$

and $12 + y_n - y_n^2 = (y_n - 3) (4 - y_n) > 0$.

I welcome other proofs for #2.

Math 424
46EV (Sec5)
9/10/19
4. Cauchy-Schwarz inequality:
\[ \left( \sum_{i=1}^{n} r_i^2 \right) \left( \sum_{i=1}^{n} s_i^2 \right) \geq \left( \sum_{i=1}^{n} r_i s_i \right)^2 \]

Apply this with \( r_i = a_i^2 \) and \( s_i = b_i^2 \):
\[ \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) \geq \left( \sum_{i=1}^{n} a_i b_i \right)^2 \]

Apply the same with \( r_i = a_i \) and \( s_i = b_i \):
\[ \left( \sum_{i=1}^{n} a_i \right) \left( \sum_{i=1}^{n} b_i \right) \geq \left( \sum_{i=1}^{n} a_i b_i \right)^2 \]

Apply these and \( r_i = a_i^2 \) and \( s_i = b_i^2 \):
\[ \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) \geq \left( \sum_{i=1}^{n} a_i b_i \right)^2 \]

Let
\[ A = \sum_{i=1}^{n} a_i^2 \quad B = \sum_{i=1}^{n} b_i^2 \quad C = \sum_{i=1}^{n} a_i^3 \quad D = \sum_{i=1}^{n} b_i^2 \]

We have from the first and third:
\[ A \cdot B \geq D^2 \Rightarrow \sqrt{AB} \geq D \]
\[ B \cdot D \geq C^2 \]

So
\[ C^2 \leq B \cdot D \leq AB = A^{3/2} B^{1/2} \]
\[ \sqrt{C^2} \leq A \cdot B^{3/2} \]

This is what we wanted!

Yes, this generalizes for other "power sums" with different exponents.

It is important that the inequalities are strict.
This comes from \( a_i^2 \neq b_i^2 \).
5. By assumption, $x$ is some kind of metric space with two "comparable" elements.

Thus,

$$d_i(x, y) < \frac{1}{50}, \text{ and } d_2(x, y) < \frac{1}{50}$$

Hence

$$y_n > 1 + \frac{n}{124y_n}$$

But this sequence $y_n$ is unbounded, a contradiction.

Next, we alternate

at $z_n = y_n^2$. Then

$$z_{n+1} = y_n^2 + \frac{1}{212} + \frac{1}{y_n^2}$$

Again, this makes $z_n$ not

$$z_{n+1} = z_n + \frac{1}{212} + \frac{1}{z_n}$$

So

$$z_n > \frac{212}{212} \rightarrow \infty$$

$$y_n > \sqrt{212}$$

The advantage here is that you get an explicit growth.

Intuition (not a theorem, as such)

Assume $y_n = f(n)$ is some smooth function.

Hence

$$y_{n+1} - y_n = \frac{1}{424y_n}$$

So

$$f(n+1) - f(n) = \frac{1}{424f(n)}$$

So you would expect an equation like

$$f'(n) = \frac{1}{424f(n)}$$

whose solution is

$$f(n) = \sqrt{n} + C$$

Not exact of course.