This may be a somewhat atypical assignment, since I don’t know yet how much you know, so I don’t know what kinds of questions are reasonable and what kinds are too hard. Recall the instructions from the Course Organization and feel free to work together in trying to understand the problems. I want you to write up the solutions individually, however and do NOT work together on any extra credit problems. The symbol (E) indicates that a problem is either an old test question or could easily have been on a test if it hadn’t been an old homework problem. If you want to prepare your solution using TeX, please put your name in your own handwriting. I never mind if you put equations in by hand.

I will answer questions about this homework in class on Wed 9/4 and Fri 9/6, and by email, but only up to Noon, Sun 9/8, so all students will have a chance to read the comments on the website, which will be finalized by 5pm 9/8.

General homework advice, based on experience from past 424 courses:
(i) Be careful in distinguishing “x < y” from “x ≤ y” in your writing.
(ii) Induction always has a basis step; be sure you are proving that “P(n) ⇒ P(n + 1)” and not “P(n + 1) ⇒ P(n)”.
(iii) It is better to write too much than to write too little. Neither the grader nor I is a psychic who can intuit what you meant to say.
(iv) It is always a good idea to write a second draft, especially if your first one has too many cross-outs. 

(ungraded) Rosenlicht §2 – #7, #12.

1. Suppose S is a bounded set in \( \mathbb{R} \) and sup(S) = M. Suppose also that a > 0 and b are real numbers. Define \( T = aS + b = \{as + b: s \in S\} \). (That is, \( T \) is the set \( S \) is scaled by a factor of \( a \) and then translated by \( b \).) Carefully show that sup(\( T \)) = aM + b. What I mean by “carefully” is that you need to show that \( aM + b \) is an upper bound and you need to show that it is a least upper bound.

2. (E) Define a sequence of real numbers as follows: \( x_0 = 12, x_{n+1} = 12 + \sqrt{x_n} \) for \( n \geq 1 \), so that \( x_1 = 12 + \sqrt{12}, x_2 = 12 + \sqrt{12 + \sqrt{12}}, \) etc. (Please note that the word “limit” does not appear in this problem; we will return to this example later in the semester.) Don’t look for a simple “closed formula” for \( x_n \).
   a. Prove by induction (or any other correct method) that for all \( n \geq 0, x_n < 16 \).
   b. Prove by induction (or any other correct method) that for all \( n \geq 0, x_n < x_{n+1} \).

3. (E) Suppose \( (X, d) \) is a metric space and suppose \( r > 0 \) and \( a \in X \). Prove from the definition of an open set that \( S = \{x : r < d(a, x) < 2r\} \) is open; that is: suppose \( y \in S \). Compute a specific \( \epsilon \) with the property that an open ball with radius \( \epsilon \) and centered at \( y \) is contained in \( S \).

Please turn over!
4. (E) Suppose \(a_i, b_i \in \mathbb{R}\). Prove that
\[
\left( \sum_{i=1}^{n} a_i^4 \right) \left( \sum_{i=1}^{n} b_i^4 \right)^3 \geq \left( \sum_{i=1}^{n} a_i b_i^3 \right)^4.
\]
Hint: It may be useful to consider versions of the Cauchy-Schwartz inequality involving the sums on the left, as well as \(\sum_{i=1}^{n} a_i^2 b_i^2\).

5. (E) Suppose \((X, d_1)\) and \((X, d_2)\) are two metric spaces with the same points and two different distances. Suppose that for all \((x, y) \in X\),
\[
3d_1(x, y) \leq d_2(x, y) \leq 5d_1(x, y).
\]
Suppose \(p \in X\). Find (with proof) a value of \(r\) so that
\[
\{ x : d_1(p, x) < r \} \subseteq \{ x : d_2(p, x) < \frac{1}{10} \}.
\]

6. (1/2 point) Extra-credit. Suppose the sequence \(\{y_n\}\) is defined by:
\[
y_0 = 1; y_{n+1} = y_n + \frac{1}{424y_n}.
\]
Show that \(\{y_n\}\) is strictly increasing (no credit), but then prove that it is **unbounded**. Definitional hint: if the sequence isn’t unbounded, there exists \(M\) so that \(y_n \leq M\) for all \(n\). Don’t look for a simple “closed formula” for \(y_n\).