1. Suppose \( \sup(S) = M \). Then

(i) \( x \in S \Rightarrow x \leq M \)

(ii) If \( x \in S \Rightarrow x \leq y \), then \( M \leq y \).

Let \( T = aS + b = \{ as + b : s \in S \} \) and \( a > 0 \).

I want to show that \( \sup(T) = aM + b \).

Let's repeat the steps:

(i) \( t \in T \Rightarrow t = as + b \), \( s \leq S \).

(ii) \( t = as + b \leq aM + b \). \( \square \)

(b) Suppose \( t \in T \Rightarrow t \leq y \). Then \( aS + b \leq y \) for all \( s \), or

\[ S \leq y/a \] for all \( s \).

Thus \( y/a \) is an upper bound for \( S \), and so by (ii),

\[ M \leq y/a \Rightarrow aM \leq y - b \Rightarrow aM + b \leq y \]

This completes the proof. There may be other ways to do this, but they will be equivalent.

2. \( x_0 = 6 \), \( x_{n+1} = 6 + \sqrt{x_n} \).

(a) To prove: For all \( n \), \( x_n < 9 \).

Base case \( x_0 = 6 < 9 \). \( \square \)

Suppose \( x_n < 9 \), then

\[ x_{n+1} = 6 + \sqrt{x_n} < 6 + \sqrt{9} = 6 + 3 = 9 \]

So the induction is immediate.

3. Suppose: \( S = \{ x : r < d(a, x) < 2r \} \)

and suppose \( y \in S \). Then \( r < d(a, y) < 2r \).

Let \( \varepsilon = \min (d(a, y) - r, 2r - d(a, y)) \).

Then \( \varepsilon > 0 \), and suppose \( 2 \in B(y, \varepsilon) \). Then

\[ d(z, y) < \varepsilon \]

Thus, \( d(a, z) \geq d(a, y) - d(y, z) > d(a, y) - \varepsilon \geq d(a, y) - (d(a, y) - r) \)

and

Calculating

\[ x_n \text{ is easy} \]

on the bottom with "next!"

Note: I feel not
expect you to do this!
4. We have by Cauchy Schwarz
\[
\left( \sum_{c=1}^{n} a_c^2 b_c \right) \left( \sum_{c=1}^{n} a_c b_c^2 \right) \\
\leq \left( \sum_{c=1}^{n} a_c^2 \right)^{\frac{1}{2}} \left( \sum_{c=1}^{n} b_c^2 \right)^{\frac{1}{2}} \\
= \left( \sum_{c=1}^{n} a_c^2 \right)^{\frac{1}{2}} \left( \sum_{c=1}^{n} b_c^2 \right)^{\frac{1}{2}}
\]
We also have
\[
\sum_{c=1}^{n} a_c^2 \geq \sum_{c=1}^{n} a_c b_c
\]
Thus,
\[
\left( \sum_{c=1}^{n} a_c^2 b_c \right)^2 \leq \left( \sum_{c=1}^{n} a_c b_c \right)^2 \\
\leq \left( \sum_{c=1}^{n} a_c \right) \left( \sum_{c=1}^{n} b_c^2 \right)
\]
as requested.

5. If \( d_1(x, y) \leq d_2(x, y) \leq 5d_1(x, y) \)

We want to make sure that
\[
d_1(p, x) < r \implies d_2(p, x) < \frac{r}{50}
\]
so looking carefully at the inequalities,
\[
d_1(p, x) < \frac{1}{50} \implies d_2(p, x) < \frac{r}{50} = \frac{r}{10}.
\]
The key's here are:
\( A \subseteq B \) means \( x \in A \implies x \in B. \)
The other inequality gives
\[
d_1(p, x) > \frac{1}{50} \implies d_2(p, x) > \frac{r}{3} \cdot \frac{1}{10},
\]
so if \( \frac{1}{50} < d_1(p, x) < \frac{1}{10}, \) then
\[
d_2(p, x) \text{ is } > \frac{r}{10} \text{ or } < \frac{r}{10},
\]
but otherwise use \( d_1. \)

6. There are (at least) three proofs I can think of.
(I). If \( m \in \{1, 2, 3, \ldots \}, \) then take
\[
e_0 = m, \ N = 0 \text{ and } m = e_0 + 1.
\]
Suppose \( |m| > 1. \) Then the integer \( n \) is defined by
\[
m = \begin{cases} 
3n+1 & m = 0 \text{ and } 3 \\
3n+2 & m = 1 \text{ and } 3 \\
3n+3 & m = 2 \text{ and } 3
\end{cases}
\]
and \( |m| \leq \frac{m+1}{3} \leq \frac{m_{\text{max}}}{3} \leq m. \)
By induction,
\[ n = \frac{\Sigma_{k=0}^{n} k^3}{k^3}, \text{ so} \]
\[ m = \frac{\Sigma_{k=0}^{n} k^3}{n} \]
\[ m = \frac{\Sigma_{k=1}^{n+1} k^3}{n+1} \]

(II) Let
\[ S_n = \sum_{k=0}^{n} k^3 \]
\[ S_0 = \{ -1, 0, 15 \} \]
\[ S_1 = \{ -1, -3, -3, -3, 0, 1, 0, 1, 2, 3, 4 \} \]
Prove by induction that
\[ S_n = \left\{ -\frac{3^{n-1}}{2}, \frac{3^{n+1}}{2} \right\} \cup \mathbb{N} \]
\[ \mathbb{Z} - \frac{3^{n+1}}{2}, -\frac{3^{n+1}}{2}, \ldots, \frac{3^{n+1}}{2}, \ldots \]
The key is not
\[ S_{n+1} = (S_n - 3^n) \cup \mathbb{N} \]
and the sets fit together perfectly. Given this, the proof is routine.

(III) If you know generating functions:
Look at
\[ (x^{-1} + x + x^2)(x^3 + x + x^2) \]
\[ \cdots + (x^{3^n} + x + x^2) \]
\[ = x^{-(1+3+\cdots+3^n)} \prod_{k=0}^{n} (1 + x^{3k} + x^{2+3k}) \]
\[ = -\left(\frac{1}{1-x} + \frac{1-x^{3n+3}}{1-x}\right) \]
\[ = \frac{1}{1-x} - \frac{1-x^{3n+3}}{1-x} \]
\[ = \frac{1}{1-x} \]

7. This used to be important
For easy proof, choose any constant.

Note that \( a > b \) if \( a - b > 0 \).

If \( a > b \), then
\[ \frac{\sqrt{a+b} - a - b}{2} = \frac{a+b-a-b}{2} > 0 \]
\[ \frac{\sqrt{a-b} - a - b}{2} = \frac{a+b-a-b}{2} = 0 \]
The same happens if \( a < b \). (Skipped)

Also, \( a > b \) if \( -b > -a \).
So, \( b = \min(a, b) = -\max(-a, -b) \).

Since this isn't graded, I'll skip the case \( a > b \).
12. \( X \cup Y = \mathbb{R}, \ X, Y \neq \emptyset \) 
\( x \in X, y \in Y \Rightarrow x < y \).
(This implies \( X \cap Y = \emptyset \) and so \( Y = x \).)
Since \( Y \) is non-empty, there exists \( y \in Y \), all \( y \) is an upper bound for \( X \).
Thus \( X \) has a least upper bound \( a \).
What do we know?
1. \( x \in X \Rightarrow x \leq a \)
2. \( x \in X \Rightarrow x \in Y \Rightarrow y \) is an upper bound for \( X \Rightarrow a \leq x \), since \( a \) is a least upper bound.
Let's clarify this and suppose \( x \neq a \). Then
\( x \in X \Rightarrow x < a \)
\( x \neq X \Rightarrow x > a \)
Since every \( x \in \mathbb{R} \) is either in \( X \) or not in \( X \), these implications go both ways:
\( x < a \Rightarrow x \in X \)
\( x > a \Rightarrow x \in X \).

(to be overproven)

If \( x < a \), \( x \leq x \), then
\( k > a \), a contradiction.
If \( x > a \), \( x \in \mathbb{R} \), then
\( x < a \), a contradiction.

If \( a \in X \), then
\( x = \{ x \in \mathbb{R} : x \leq a \} \)
\( = \{ x \in \mathbb{R} : x < a \} \cup \{ a \} \).
If \( a \in X \), then
\( x = \{ x \in \mathbb{R} : x < a \} \).

**Bonus Half-Note.**

If \( E = \mathbb{R}^n \), \( d(x) \) is defined as follows:
\[ p = (x_1, \ldots, x_n) \]
\[ q = (y_1, \ldots, y_n) \]
\[ d(x, p) = \max \{ |x_i - y_i| \} \]  
\( 1 \leq i \leq n \)

Considering (1), (2), (3) are understandable to show (4), let \( r = (z_1, \ldots, z_n) \).
Then
\[ d(x, p) = \max \{ |x_i - z_i| \} \]  
\( 1 \leq i \leq n \)
\[ = |x_j - z_j| \text{ for some } j \]
\[ \leq |x_j - y_j| + |y_j - z_j| \text{ (usual triangle inequality) } \]
\[ \leq \max \{ |x_i - y_i| \} + \max \{ |y_i - z_i| \} \]  
\( 1 \leq i \leq n \)
\[ = d(x, p) + d(x, q, r) \]