This homework is written without feedback from the first assignment. Recall the instructions from the Course Organization and feel free to work together in trying to understand the problems. I want you to write up the solutions individually, however, and do **NOT** work together on any extra credit problems. (No extra credit this week.) The symbol ($\mathcal{E}$) indicates that a problem is either an old test question or could easily have been on a test if it hadn’t been an old homework problem. If you want to prepare your solution using TeX, please put your name in your own handwriting. I never mind if you put equations in by hand.

I will answer questions about this homework in class on Wed. 9/7 and Fri. 9/9 and by email, but only up to Noon, Sun. 9/11, so all students will have a chance to read the comments on the website, which will be finalized by 3pm 9/11.

(ungraded) *Rosenlicht* §3 – #2, #6.

1. ($\mathcal{E}$) Recall that the following metrics defined in class for $X = \mathbb{R}^n$ for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$:

   $$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \quad d_2(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}, \quad d_\infty(x, y) = \max_i \{|x_i - y_i|\}.$$  

   a. Prove that the following inequalities hold for $x, y \in \mathbb{R}^n$:

   $$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n}d_\infty(x, y)$$  

   (Hint: you do not need Cauchy-Schwartz!)

   b. Find $z \in \mathbb{R}^n$ so that $d_2(z, 0) = \sqrt{n} \cdot d_\infty(z, 0)$ and $d_1(z, 0) = \sqrt{n} \cdot d_2(z, 0)$. A little experimentation is all you need.

2. ($\mathcal{E}$) Using the ($\epsilon, N$) definition of limit, prove that

   $$\lim_{n \to \infty} \sqrt{9 - \frac{1}{n^2}} = 3.$$ 

   (Please note that for this problem, no other methods are acceptable!)

3. ($\mathcal{E}$) Suppose $(X, d)$ is a metric space and $p_1 \neq p_2$ are two points in $(X, d)$. Construct open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in $(X, d)$ such that $p_1 \in \mathcal{O}_1$, $p_2 \in \mathcal{O}_2$ and $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. Explain how you have satisfied each of the desired criteria.

4. ($\mathcal{E}$) Suppose $(x_n)$ is a Cauchy sequence in a metric space $(X, d)$ and suppose $(y_n)$ is a sequence with the property that $\lim_{n \to \infty} d(x_n, y_n) = 0$. Prove from the definition that $(y_n)$ is also a Cauchy sequence. (Hint: There is no assumption that $(X, d)$ is complete.) The first sentence of your solution should be: “Suppose $\epsilon > 0$ is given.”

Please turn over!
5. (E for a,b,c, not d.) Let $f(x) = 2 + \frac{3}{x}$ and define a sequence $(a_n)$ by $a_0 = 2$ and $a_n = f(a_{n-1})$ for $n \geq 1$.

a. Prove that $a_n \geq 2$ for all $n \geq 0$.

b. Let $b_n = 3 - a_n$. Find a function $g$ so that $b_n = g(b_{n-1})$.

c. Prove that $\lim_{n \to \infty} a_n = 3$ by using b. (and a. in the form $b_n \leq 1$).

d. Find a “closed formula” for $a_n$. The answer I have in mind uses several instances of $k^n$, where $k$ is a specific small positive integer.