1. Rosenlicht Ch. V-8.

2. Rosenlicht Ch. V-14.

3. (E) True or false. If these statements are true, give a short proof. If these statements are false, give a simple counterexample with explanation. A sufficiently clear sketch will do. Read the problems carefully!

a. If \( f \) is a function defined on \((0, 1)\) and \( f(x) < \frac{1}{x} \) for all \( x \in (0, 1) \), then \( f \) is unbounded.

b. If \( f : \mathbb{R} \mapsto \mathbb{R} \) is continuous on \([-1, 1]\) and \( f(\frac{1}{2}) \geq f(x) \geq f(-\frac{1}{2}) \) for all \( x \in [-1, 1] \), then \( f \) must be differentiable at either \( \frac{1}{2} \) or \( -\frac{1}{2} \).

c. There exists a real number \( r \) with the property that \( 0 < r < 1 \) and \( r^{13} - 4r^5 + 2 = 0 \).

4. (E) Suppose \( f(x) \) is defined and continuous on \((a, b)\) and \( g(x) \) is defined and continuous on \([b, c]\) and suppose further that \( f(b) = g(b) \). Let \( \Phi(x) \) be defined so that if \( x \in (a, b) \), then \( \Phi(x) = f(x) \), and if \( x \in [b, c] \), then \( \Phi(x) = g(x) \). Prove carefully that \( \Phi(x) \) is continuous on \((a, c)\). The point \( x_0 = b \) requires special attention.

5. (E) Suppose \( f(x) \) is a function defined on \([-1, 1]\) and \( |f(x)| < 17 \) for all \( x \). You do not know that \( f \) is necessarily continuous, just that it is bounded.

a. Prove carefully that \( g(x) = xf(x) \) must be continuous at \( x_0 = 0 \).

b. Prove carefully that \( h(x) = 3x + x^2 f(x) \) must be differentiable at \( x_0 = 0 \) and \( h'(0) = 3 \).

6. Recall the Mean Value Theorem: if \( f \) is differentiable on an open interval containing \( a < b \), then there exists \( c \) so that \( a < c < b \) and

\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

a. Suppose \( f(x) = c_0 x^2 + c_1 x + c_2 \). Express \( c \) as an explicit function of \((a, b)\), independent of \( c_0, c_1, c_2 \).

b. (Extra credit, 1/2 point). Find all functions \( g : \mathbb{R} \mapsto \mathbb{R} \) with the property that \( g, g', g'' \) exist and, for all \( a, b \in \mathbb{R} \)

\[
\frac{g(b) - g(a)}{b - a} = g' \left( \frac{a + b}{2} \right).
\]

(Possible hint: consider \( b = t + u, a = t - u \) and plug in and keep differentiating! You may use elementary facts about antiderivatives.)

c. (Extra extra credit, 1/2 point). Exact same problem with

\[
\frac{g(b) - g(a)}{b - a} = g' \left( \frac{2a + b}{3} \right).
\]

(No hints.)