The maximum possible score on Problem 6 is 1\frac{1}{2} points, so it is partially a regular question and partially extra credit.

1. *Rosenlicht* Ch. IV – 16.

2. *Rosenlicht* Ch. IV – 33.


4. (E) Suppose \( f : [0, 1] \rightarrow [0, 1] \) is a continuous function. Show that there exists \( x \in [0, 1] \) so that \( f(x) = x \). Hint: what can you say about \( g(x) = f(x) - x, \ g(0) \) and \( g(1) \)? Remark: this is a classical type of exam problem, which appears with variations on several homeworks as well, without hints in the future!

5. (E) Let

\[
f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k(k+1)}.
\]

Prove, from the definition, that \( (f_n) \) converges uniformly to a continuous function \( f \) on \([-1, 1]\). As a hint for estimating \( f_n(x) - f_m(x) \), note that \( \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \) and that \( |x^k| \leq 1 \) on \([-1, 1]\).

6. Partially extra credit, worth 1\frac{1}{2} points.

Suppose \( f([0, 1]) = [0, 1] \) is onto and \( f \) is monotone non-decreasing; that is, \( x < y \) implies \( f(x) \leq f(y) \). One goal of this problem is to prove that a monotone function without “jumps” must be continuous, and another is to give an alternate proof of the continuity of functions like the Cantor function.

(i) Prove that \( f(0) = 0 \) and \( f(1) = 1 \). (This is easy if you think about it.)

(ii) Suppose \( u \in (0, 1) \) and \( \epsilon > 0 \) is given. Prove that there exists \( \delta > 0 \) so that \( |x - u| < \delta \) implies that \( f(u) - \epsilon < f(x) < f(u) + \epsilon \). Hint: because \( f \) is onto, there exist \( v, w \) such that \( f(v) = \max\{0, f(u) - \epsilon/2\} \) and \( f(w) = \min\{1, f(u) + \epsilon/2\} \).