1. Rosenlicht Ch. III #10. (Think about cluster points.)

2. Rosenlicht Ch. III #11. (As a hint, let \( b_n = \frac{1}{n} \cdot \sum_{i=1}^{n} a_i \), and let \( \lim_{i \to \infty} a_i = L \). To help get you started: if \( \epsilon > 0 \) is given, then there exists \( N_1 \) so that \( n > N_1 \) implies that \( |a_i - L| < \epsilon/2 \).

b. (Extra credit 1/4 point) Find a sequence \((a_n)\) which is not convergent, but for which \((b_n)\), as defined above, is convergent. Prove all your assertions.

3. \((X)\) Suppose \( f \) is a bijection of \( \{0, 1, 2, \ldots\} \) to itself. Suppose \((a_n)\) is a sequence in a metric space \((X, d)\). We define the \textit{rearrangement} of \( a_n \) by \( f \) to be the sequence \((b_n)\) where \( b_n = a_{f(n)} \).

One example of a rearrangement of \((a_n)\) is \( a_1, a_0, a_3, a_2, a_5, a_4, \ldots \). If \((a_n)\) is convergent and \( \lim_{n \to \infty} a_n = L \), prove carefully that \((b_n)\) is convergent and \( \lim_{n \to \infty} b_n = L \).

4. \((X)\) Suppose \( A \) and \( B \) are non-empty bounded subsets of \( \mathbb{R} \). The \textit{Minkowski sum} of \( A \) and \( B \) is defined to be \( C = \{ c : \text{there exist } a \in A, b \in B \text{ with } c = a + b \} \).

Prove carefully that \( \sup C = \sup A + \sup B \).

5. \((X)\) Recall \( d_1, d_2, d_\infty \) defined in class for \( X = \mathbb{R}^n \): \( d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| \), \( d_2(x, y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{1/2} \), \( d_\infty(x, y) = \max_i \{|x_i - y_i|\} \).

a. Prove that each of the following two inequalities hold:

\[
d_\infty(x, y) \leq d_2(x, y), \quad d_2(x, y) \leq d_1(x, y)
\]

Note: these are easier than the ones from HW2, #4. In the second case, I would square the inequality, think about the cross-terms, then work backward.

b. The heart of this problem: Using your results in a. and HW2, #4, prove that (for fixed \( n \)), \( S \) is an open set in \((\mathbb{R}^n, d_1)\) if and only if it is an open set in \((\mathbb{R}^n, d_2)\) if and only if it is an open set in \((\mathbb{R}^n, d_\infty)\).

6. Let \( f(x) = 2 + \frac{3}{x} \) and define a sequence \((a_n)\) by \( a_0 = 2 \) and \( a_n = f(a_{n-1}) \) for \( n \geq 1 \).

a. Prove that \( a_n \geq 2 \) for all \( n \geq 0 \).

b. Let \( b_n = 3 - a_n \). Find a function \( g \) so that \( b_n = g(b_{n-1}) \).

c. Prove that \( \lim_{n \to \infty} a_n = 3 \) by using b. (and a. in the form \( b_n \leq 1 \)).

\d. (Extra credit, 1/4 pt.) Find a closed formula for \( a_n \). The answer I have in mind uses several instances of \( k^n \), where \( k \in \mathbb{Z} \) and \( |k| \) is small.