This is a somewhat atypical assignment, since we won’t be talking much about fields other than \( \mathbb{R} \) after this week. Recall the instructions and feel free to work together in trying to understand the problems. I want you to write up the solutions individually, however. I will be answering email questions over the Labor Day Weekend, as long as the rains don’t knock out power to my router.

1. *Rosenlicht* §2 #1. (Hint: Let the field elements be “0, 1, \( x \)” and try to write consistent addition and multiplication tables.)


5. Recall that \( \mathbb{Q} \) denotes the field of rational numbers. Suppose \( u \notin \mathbb{Q} \) and

\[
K = \{ a + b \cdot u \mid a, b \in \mathbb{Q} \}
\]

is a field. (One example of such a field occurs when \( u = \sqrt{2} \).) Prove that there exist rational numbers \( c, d \) and an integer \( m \) which is not a perfect square so that \( u = c + d \cdot \sqrt{m} \). Find \( r, s \), which will depend on \( c, d, m \), so that \( 1/u = r + s \cdot u \).

Hint: Since \( K \) is a field and \( u \in K \), we know that \( u \cdot u \in K \) as well.

6. Recall that \( \sqrt{2} \notin \mathbb{Q} \). For purposes of this problem, \( \sqrt{2} \) is the unique positive real number such that \((\sqrt{2})^2 = 2\). There is no reason to think about its decimal expansion 1.414..., which should not appear in your solution!

Let \( A = \{ x \in \mathbb{Q} : x < \sqrt{2} \} \) and let \( B = \{ x \in \mathbb{Q} : x > \sqrt{2} \} \), so that \( A \cup B = \mathbb{Q} \).

a. Suppose \( y \in B \) and let \( z = \frac{y^2 + 2}{2y} \). Prove that \( y > z > \sqrt{2} \).

b. Suppose \( u \in A \) and let \( v = \frac{4u}{u^2 + 2} \). Prove that \( \sqrt{2} > v > u \).

c. Show that 2 is an upper bound for \( A \). (Possible hint: suppose it isn’t an upper bound!)

d. Use the first three parts of this problem to show that the field \( \mathbb{Q} \) fails to satisfy the least upper bound property (Property VII *Rosenlicht*, p. 24), because the set \( A \) is non-empty and bounded above, but has no least upper bound within \( \mathbb{Q} \).