Math 417
HW7
Due 3/15/19

§1.14-3 In $(\mathbb{Z}/4\mathbb{Z})\times(\mathbb{Z}/2\mathbb{Z})$, the element $(2,1)$ has order 2:
$\langle(2,1)\rangle = \{(0,0), (2,1), (4,2)=(0,0)\}$
so the order of the factor group is $\frac{4\times 2}{2}=4$

§1.14-5 The element $(1,1)$ has order 4
$\{(0,0),(1,1),(0,2),(1,3)\}$
in $(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/4\mathbb{Z})$, so the order of the factor group is $\frac{2\times 4}{4}=2$.

§1.14-9. $\mathbb{Z}/12\mathbb{Z}/(4)$ is the factor group of $\{[0]_{12},[4]_{12},[8]_{12}\}$. Here are the cosets
$\{[0]_{12},[4]_{12},[8]_{12}\}$
$\{[1]_{12},[5]_{12},[9]_{12}\}$
$\{[2]_{12},[6]_{12},[10]_{12}\}$
$\{[3]_{12},[7]_{12},[11]_{12}\}$
$5+\langle 4\rangle$ is $\{[1]_{12},[5]_{12},[9]_{12}\}$ and this has order 4, as the quotient group is cyclic of order 4, and $5+\langle 4\rangle$ is a generator

Graded Problems

1a §1.13-22
$\phi:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ operation is "+"
$\phi(1,0)=3 \quad \phi(0,1)=-5$
so $\phi(m,n)=\phi(m,0)+\phi(0,n)$
$=m\phi(1,0)+n\phi(0,1)=$
$3m-5n$
$\ker\phi=\{(m,n): 3m-5n=0\}$
$3m=5n \Rightarrow 3\mid 5n \Rightarrow 3\mid n$
($\gcd(3,5)=1$), so $n=3t$ and $m=5t$ and so
$\ker\phi=\{(5t,3t): t\in\mathbb{Z}\}$.
$\phi(-3,2)=-3\phi(1,0)+2\phi(0,1)=-3\cdot 3+2\cdot 5=-9+10=1.$

1b. §1.14-2
So $|\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/12\mathbb{Z}|=4\cdot 12=48.$
In $\mathbb{Z}/4\mathbb{Z}$, $\langle[2]_4\rangle=\{[0]_4,[2]_4\}$ and $|\langle[2]_4\rangle|=2$
In $\mathbb{Z}/12\mathbb{Z}$, $\langle[2]_{12}\rangle=\{[0]_{12},[2]_{12},\dots,[10]_{12}\}$ and $|\langle[2]_{12}\rangle|=6$, so the order of $\langle[2]_4\rangle\times\langle[2]_{12}\rangle$ is $2\cdot 6=12$ and the order of the factor group is $\frac{48}{12}=4$. If you work it out (not asked), the elements of the factor group are {(even, even), (even, odd), (odd, even), (odd, odd)}

2. So, fix a, let $\Phi_a(x)=a^{-1}xa$ map $G\to G$.
(1) is it 1-1? $a^{-1}xa=a^{-1}ya$
$\Rightarrow a(a^{-1}xa)a^{-1}=a(a^{-1}ya)a^{-1}\Rightarrow$
$x=y$ yes ✓
(2) is it onto? $\Phi_a(x)=y \Rightarrow a^{-1}xa=y$
$\Rightarrow x=aya^{-1}$, so $\Phi_a(aya^{-1})=x$
yes it's onto ✓
(3) Does $\Phi_a(xy)=\Phi_a(x)\Phi_a(y)$?
$a^{-1}(xy)a \overset{?}{=} (a^{-1}xa)(a^{-1}ya)$
$\overset{?}{=} a^{-1}x(aa^{-1})ya$
$\overset{?}{=} a^{-1}xya$ ✓

So Φ (called conjugation by a) does all three and is an isomorphism.
Note: If G is abelian, then $\Phi_a(x)=x$!

3a $G = \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/42\mathbb{Z}$.

if $a = ([m]_{16}, [n]_{42})$, then the order of a

$= \mathrm{lcm}(\underset{\mathbb{Z}/16\mathbb{Z}}{\mathrm{ord}\,[m]_{16}}, \underset{\mathbb{Z}/42\mathbb{Z}}{\mathrm{ord}\,[n]_{42}})$

if you want it to be 28, one way is $\mathrm{ord}\,[m]_{16} = 4$ $\mathrm{ord}\,[n]_{42} = 7$

(another is $\mathrm{ord}\,[m]_{16} = 4$, $\mathrm{ord}\,[n]_{42} = 14$.

I'll give examples of each

$\mathrm{ord}\,[\frac{16}{4}]_{16} = \mathrm{ord}\,[4]_{16} = 4$

$\mathrm{ord}\,[\frac{42}{7}]_{42} = \mathrm{ord}\,[6]_{42} = 7$

so $\mathrm{ord}\,([4]_{16}, [6]_{42}) = 28$

There are lots of choices.

similarly $\mathrm{ord}\,([4]_{16}, [3]_{42})$

$= \mathrm{lcm}\,(4, 14) = 28$.

G has $16 \cdot 42 = 672$ elements

That's why I didn't ask for all

3.b Answer 1. The order of $[m]_{16}$ divides 16; the order of $[n]_{42}$ divides 42, so neither has a factor of 5.

Answer 2, Suppose $5a = 0$.

For $a = ([m]_{16}, [n]_{42})$. Then

$5m \equiv 0 \bmod 16 \Rightarrow 16 \mid 5m \Rightarrow 16 \mid m$

$5n \equiv 0 \bmod 42 \quad 42 \mid 5n \Rightarrow 42 \mid n$

because $\gcd(5,16) = \gcd(5,42) = 1$.

That is, $a = ([m]_{16}, [n]_{42})$

$= 0 = ([0]_{16}, [0]_{42})$ actually, so a has order 1, not 5

4. Suppose $\phi : V \to (\mathbb{Z}/7\mathbb{Z}, +)$

We know $\phi(e) = [0]_7$ because ϕ is a homomorphism

Let $\phi(x) = [a]_7$. Then $\phi(x^2)$

$= [2a]_7$, because ϕ is a homomorph.

But $x^2 = e$, so $\phi(e) = [2a]_7 \Rightarrow$

$= [0]_7$

$2a \equiv 0 \bmod 7$

$\Rightarrow a \equiv 0 \bmod 7$. That is,

$\phi(x) = [0]_7$. The same argument applies to y, z because $\phi(y^2) = \phi(e)$ and $\phi(z^2) = e$. Thus ϕ is the boring homomorphism: $\phi(e) = \phi(x)$

$= \phi(y) = \phi(z) = [0]_7$.

5,6. $G = (\mathbb{Z}/6\mathbb{Z}, +)$ $H = C_{12} = \langle a \rangle = \{e, \ldots a^{11}\}$

ϕ is defined by $\phi([1]_6) = a^4$, so

$\phi([2]_6) = a^8$, $\phi([3]_6) = a^{12} = e$, $\phi([4]_6) = a^{16} = a^4$, $\phi([5]_6) = a^{20} = a^8$

a). Reading this, $\ker \phi = \{[0]_6, [3]_6\}$

b) Reading this $\mathrm{im}\,\phi = \{e, a^4, a^8\}$.

c) So $K = \{0, 3\}$ (I'll drop the $[\,]_6$)

It's easy to compute the cosets, $1 + K = \{1, 4\}$, $2 + K = \{2, 5\}$ — remember the operation is addition.

Here is the "multiplication" table

	$\{0,3\}$	$\{1,4\}$	$\{2,5\}$
$\{0,3\}$	$\{0,3\}$	$\{1,4\}$	$\{2,5\}$
$\{1,4\}$	$\{1,4\}$	$\{2,5\}$	$\{0,3\}$
$\{2,5\}$	$\{2,5\}$	$\{0,3\}$	$\{1,4\}$

because, eg, $\{1,4\} + \{2,5\} =$

$\{1+2, 1+5, 4+2, 4+5\} = \{3, 6, 6, 9\}$

$= \{3, 0, 0, 3\} = \{0, 3\}$.

Note: $\{0,3\} = \phi^{-1}(e)$, $\{1,4\} = \phi^{-1}(a^4)$

$\{2,5\} = \phi^{-1}(a^8)$