Ungraded

Math 417 HW5 Due 3/1/19

§1.9-9 $(1,2)(4,7,8)(2,1)(7,2,8,1,5)$

$1 \to 5$
$2 \to 8 \to 4$
$3 \to 3$ (unchgd)
$4 \to 7$
$5 \to 7 \to 8$
$6 \to 6$ (unchgd)
$7 \to 2 \to 1 \to 2$
$8 \to 1 \to 2 \to 1$

so
$1 \to 5 \to 8 \to 1$
$2 \to 4 \to 7 \to 2$
$3 \to 3$
$6 \to 6$
$(158)(247)(3)(6)$.
or $\begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 5&4&3&7&8&6&2&1 \end{pmatrix}$

§1.9-11 $1 \to 3 \to 4 \to 1$ $5 \to 8 \to 7 \to 5$
$2 \to 6 \to 2$ so $(134)(26)(587)$
$(134) = (14)(13)$, $(587) = (57)(58)$
so it's also $(14)(13)(26)(57)(58)$

§1.10-1 $4\mathbb{Z} = \{-8, -4, 0, 4, 8, 12 \ldots\}$
so $4\mathbb{Z}$ is one of the cosets, the others are $1+4\mathbb{Z} = \{-7, -3, 1, 5, 9 \ldots\}$, $2+4\mathbb{Z}$ and $3+4\mathbb{Z}$. The cosets are also called $[0]_4, [1]_4, [2]_4, [3]_4$

§1.10-3 $\mathbb{Z}_{12} = \{[0]_{12}, [1]_{12}, \ldots, [11]_{12}\}$
The subgroup $\langle [2]_{12} \rangle$ is
$\{[0]_{12}, [2]_{12}, [4]_{12}, \ldots [10]_{12}\}$
The only other coset is $1 + \langle [2]_{12} \rangle$
$= \{[1]_{12}, [3]_{12}, [5]_{12}, \ldots, [11]_{12}\}$

1\. §1.8-40. ~~A is a finite set and $b \in A$ is a fixed element.~~
S_A = group of permutations of A
$G = \{\sigma \in S_A : \sigma(b) = b\}$.
To prove G is a subgroup:
(1) Is G closed?
$\sigma_1 \in G$, $\sigma_2 \in G$. Then $\sigma_1(b) = b$, $\sigma_2(b) = 2$. Is $\sigma_1 \sigma_2 \in G$?
$(\sigma_1 \sigma_2)(b) = \sigma_1(\sigma_2(b)) = \sigma_1(b) = b$.
Yes, G is closed.

(2). Is $e \in G$?
Sure. If e is the identity permutation then $e(b) = b$.
(3) If $\sigma \in G$, is $\sigma^{-1} \in G$?
Well σ^{-1} is a permutation of S_A and $\sigma(x) = y \Rightarrow \sigma^{-1}(y) = x$, so $\sigma(b) = b \Rightarrow \sigma^{-1}(b) = b$, i.e. $\sigma^{-1} \in S_A$.
The answer is yes: all three conditions

2\. §1.9-34
Let $\sigma = (x_1, \ldots, x_{2n+1})$ odd length
so $\sigma(x_1) = x_2$, $\sigma(x_2) = x_3, \ldots, \sigma(x_{2n+1}) = x_1$.
What is σ^2. $\sigma^2(x_1) = \sigma(\sigma(x_1)) = \sigma(x_2) = x_3$
$\sigma^2(x_2) = x_4 \ldots$ up $\sigma^2(x_{2n}) = \sigma(x_{2n+1}) = x_1$
and $\sigma^2(x_{2n+1}) = x_2$
so $x_1 \to x_3 \to \ldots \to x_{2n+1} \to x_2 \to x_4 \ldots \to x_{2n} \to x_1$
I'd say that's a cycle!
(Note: why odd length? Because if even length, then σ^2 splits into a product of two cycles of equal length.
eg. $(x_1 x_2 x_3 x_4 x_5 x_6)^2 = (x_1 x_3 x_5)(x_2 x_4 x_6)$)

3\. Suppose G is a group and $\Phi(x) = x^{-1}$ is an automorphism of G; that is, Φ is an isomorphism of G to G.
Then $\Phi(xy) = \Phi(x)\Phi(y)$, so
$(xy)^{-1} = x^{-1} \cdot y^{-1}$
$\Rightarrow (xy)^{-1} \cdot y = (x^{-1} \cdot y^{-1}) \cdot y = x^{-1}(y^{-1}y) = x^{-1}$
$\Rightarrow (xy)^{-1} \cdot yx = x^{-1} \cdot x = e$
so $(yx) \cdot (xy)^{-1} = e$
But $(xy)(xy)^{-1} = e$, so $\underline{xy = yx}$
G is abelian
(There are other, equally valid, ways to do the symbolic manipulation to get $xy = yx$.)

4. $G = D_4 = \{(1)(2)(3)(4), (1234), (13)(24), (1432), (12)(34), (14)(23), (1)(24)(3), (13)(2)(4)\}$

$H_1 = \{(1)(2)(3)(4), (12)(34)\}$.

We follow the class approach. Pick elements that we haven't seen and use them to find the next coset.

Left cosets

$(1)(2)(3)(4)H_1 = \{(1)(2)(3)(4), (12)(34)\}$ *

I don't see (1234).

$(1234)H_1 = \{(1234)(1)(2)(3)(4), (1234)(12)(34)\}$

$= \{(1234), (13)(2)(4)\}$ *

because $(1234)(12)(34)$: $1\to2\to3$, $2\to1\to2$, $3\to4\to1$, $4\to3\to4$, etc.

I don't see (13)(24).

$(13)(24)H_1 = \{(13)(24)(1)(2)(3)(4), (13)(24)(12)(34)\}$

$= \{(13)(24), (14)(23)\}$. *

I don't see (1432).

$(1432)H_1 = \{(1432)(1)(2)(3)(4), (1432)(12)(34)\}$

$= \{(1432), (1)(24)(3)\}$ *

The underlined + starred sets are the left cosets. In exactly the same way, the right cosets are

$H_1(1)(2)(3)(4) = \{(1)(2)(3)(4), (12)(34)\}$ **

$H_1(1234) = \{(1234), (1)(24)(3)\}$ **

$H_1(13)(24) = \{(13)(24), (14)(23)\}$ **

$H_1(1432) = \{(1432), (13)(2)(4)\}$

Note that two of the left cosets are also right cosets, but two aren't. So it goes.

5. As usual, we need to check closure, identity and inverses.

If $x \in H$ then $x = g^2$ for $g \in G$. So if $x_1, x_2 \in H$, then $x_1 x_2 = g_1^2 g_2^2 = (g_1 g_1)(g_2 g_2) = g_1(g_2 g_1)g_2 = (g_1 g_2)^2$ because G is abelian.

Also, $e = e^2$, so $e \in H$ and if $x = g^2 \in$ then $x^{-1} = (g^{-1})^2$ $[(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}]$, so $x^{-1} \in G$. Thus H is a subgroup.

6. What does it mean to say that $aH = Ha$? If $h_1 \in H$, then $a \cdot h_1 = h_2 \cdot a$ for some $h_2 \in H$. In other words, if $h_1 \in H$, then $h_2 = a h_1 a^{-1}$ is also in H.

In other words $(aH)a^{-1} = (Ha)a^{-1} = H(aa^{-1})$, or $aHa^{-1} = H$.

6a. Suppose $a \in H$. Then $aH = H$ and $Ha = H$, so $aH = Ha$ and $a \in K$.

6b. Again: closure, identity, inverses.

Suppose $a, b \in K$. Then

$(ab)H = a(bH) = a(Hb)$ $(b \in K)$

$= (aH)b$ (associativity)

$= (Ha)b$ $(a \in K)$.

Inverses can be tricky, but

$a \in K \Rightarrow aH = Ha$

$\Rightarrow (aH)(a^{-1}) = (Ha)a^{-1} = H(aa^{-1})$

$\Rightarrow aHa^{-1} = H$

$\Rightarrow a^{-1}(aHa^{-1}) = a^{-1}H$

$= (a^{-1}a)Ha^{-1} = a^{-1}H$

$\Rightarrow Ha^{-1} = a^{-1}H$

$\Rightarrow a^{-1} \in K$

So, yes, a subgroup