(2) If \( e \in G \), sure, \( e \) is the identity permutation. Then \( e \) is iden.

(3) If \( e \in G \), then \( e \) is in \( S_5 \).

Well \( 0^4 \) is a permutation of \( S_5 \) and \( T(x) = y \) or \( T^4(x) = x \), so \( T(6) = 6 \Rightarrow T^4(6) = 6 \), i.e., \( 6 \in S_5 \).

The answer is yes: all three are.

2. \( e \in S_4 \)

Let \( \sigma = (x_1, x_2) \) odd length.

So \( \sigma(x_1) = x_2 \), \( \sigma(x_2) = x_3 \), ... \( \sigma(x_{n-1}) = x_n \).

What is \( \sigma^2 \). \( \sigma^2(x_1) = \sigma(\sigma(x_1)) = \sigma(x_2) = x_3 \)
(\( \sigma^2(x_2) = x_4 \) ... \( \sigma^2(x_{n-1}) = x_{n-1} \)
\( \sigma^2(x_n) = x_1 \)

It's \( \sigma(x_{n+1}) = x_2 \)

So \( x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_{n-1} \rightarrow x_n \)

I'd say that's a cycle.

Check: why odd length? Because of even length, then \( \sigma^2 \) splits into a product of two cycles of equal length.

E.g., \((x_1 \ x_2 \ x_3 \ x_4 \ x_5) \ x_6 = (x_1 \ x_3 \ x_5)(x_2 \ x_4 \ x_6) \)

3. Suppose \( G \) is a group and \( \sigma \in S \) a one-to-one mapping of \( G \),

\( \Phi(x) = x^{-1} \) is an automorphism of \( G \);

That is, \( \Phi \) is an isomorphism of \( G \).

Then \( \Phi(xy) = \Phi(x) \Phi(y) \), so

\( (xy)^{-1} = x^{-1} y^{-1} \)

\( \Rightarrow (xy)^{-1} y = (x^{-1} y^{-1} y) x^{-1} \)

\( \Rightarrow (xy)^{-1} y = x^{-1} x = e \)

So \( (xy)^{-1} = e \)

But \( (xy) (xy)^{-1} = e \), so \( xy = yx \)

Therefore, for any \( \Phi \), \( G \) is a group valid, to be representative of a group, to get \( xy = yx \).
4. \( G = D_4 = \langle (1 2 3 4), (1 2 3 4) \rangle \Rightarrow \langle (1 2 3 4), (1 2)(3 4) \rangle \Rightarrow \langle (1 2)(3 4), (1 4)(2 3) \rangle \rangle \)

- We follow the class approach. First, elements that we haven't seen add up to find the next coset.

**Left cosets**

\((1 2 3 4)H = \{ (1 2 3 4), (1 2)(3 4) \} \) \( \times \)

I didn't see \((1 2 3 4)\).

\((1 2 3 4)H = \{ (1 2 3 4), (1 2)(3 4) \} \) \( \times \)

because \((1 2 3 4)(1 2)(3 4) = \{ 1 \} \times 2 \times 3 \times 4 = 1 \)

I didn't see \((1 2)(3 4)\).

\((1 2)(3 4)H = \{ (1 2)(3 4), (1 4)(2 3) \} \) \( \times \)

I didn't see \((1 4)(2 3)\).

\((1 4)(2 3)H = \{ (1 4)(2 3), (1 2)(3 4) \} \) \( \times \)

The individual internal sets are the left cosets. In exactly Rosanne's,

**Right cosets**

\( H, \{ (1 2)(3 4) \}, \{ (1 2)(3 4) \}, \{ (1 2)(3 4) \} \)

\( H, \langle (1 2)(3 4) \rangle, \langle (1 2)(3 4) \rangle, \langle (1 2)(3 4) \rangle \) \( \times \)

\( H, \langle (1 2)(3 4) \rangle = \{ (1 2)(3 4), (1 2)(3 4) \} \times \)

\( H, \langle (1 2)(3 4) \rangle = \{ (1 2)(3 4), (1 2)(3 4) \} \times \)

Note: Only the left cosets are also right cosets, but two aren't. So it goes.

5. As usual, we need to check closure, identity, and inverses.

6. What does it mean to say that \( aH = Ha \)? If \( aH \), then \( aH h = a \).

In other words, if \( h \in H \), then \( h = aH \).

In other words, \( aH = Ha \).

6a. Suppose \( a \in H \). Then \( aH = Ha \).

6b. Again: closure, identity, inverses

Suppose \( a, b \in K \). Now

\( (ab)H = a(Hb) = a(Hb) \) \( \times \)

\( = (aH)b \) \( \times \) (associativity)

\( = (aH)b \)

Inverses can be tricky, so

\( a \in K \Rightarrow aH = Ha \Rightarrow aH(aH) = (Ha)a \Rightarrow aH = Ha \Rightarrow aH(aH) = aH \Rightarrow a^{-1} = a^{-1} \)

So, yes, a subgroup.