Math 417
HW 4
Due 2/15/19

Ungraded

§1.6 - 46 - Suppose $a, b \in G$ and $(ab)^n = e$. Then, as noted,
$a(ba)^{n-1}b = e$
Now multiply on the left by a^{-1} and on the right by b^{-1}.

$$\underbrace{a^{-1} \cdot a}_{e}(ba)^{n-1} \cdot \underbrace{b \cdot b^{-1}}_{e} = \underbrace{a^{-1} \cdot b^{-1}}_{=(ba)^{-1}}$$

$\Rightarrow (ba)^{n-1} = (ba)^{-1}$.
Now multiply both sides by (ba).

$$\underbrace{(ba)^{n-1}(ba)} = (ba)^{-1}(ba) = e$$

$$(ba)^n = e.$$

§1.8 - 1, 3

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$ OR $(1\,3\,4\,5\,6\,2)$

$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$ $(1\,2\,4\,3)(5\,6)$

$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{pmatrix}$ $(1\,5)(2)(3\,4)(6)$

$\tau\sigma$: $\cancel{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}}$

~~$1 \to 3 \to 1$~~
~~$2 \to 1 \to 2$~~
~~$3 \to 4 \to 3$~~
$4 \to 5 \to 6$
~~$5 \to 6 \to 5$~~
$6 \to 2 \to 4$

$\cancel{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix}}$ ~~OR $(1)(2)(3)(4\,6)(5)$~~

$\mu\sigma^2$. First ~~the~~ σ^2.

$1 \to 3 \to 4$ $\quad 3 \to 4 \to 5$ $\quad 5 \to 6 \to 2$
$2 \to 1 \to 3$ $\quad 4 \to 5 \to 6$ $\quad 6 \to 2 \to 1$

$\sigma^2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 2 & 1 \end{pmatrix}$

$\mu\sigma^2$. $\quad \sigma^2 \quad \mu$ $\qquad$ Or $(1\,4\,6)(2\,5)$

$1 \to 4 \to 3$
$2 \to 3 \to 4$
$3 \to 5 \to 1$
$\to 6 \to 6$
$\to 2$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 6 & 2 & 5 \end{pmatrix}$ or $(1\,3)(2\,4\,6\,5)$

Graded

1a §1.5 - 43
Suppose G is an abelian group. H, K are subgroups of G.
~~Prove that $HK = \{hk : h \in H, k \in K\}$ is a subgroup of G.~~

By Thm 5.14, I have to prove that
HK ~~is closed~~
(1) is closed
(2) has the identity element
(3) has inverses.

I will use that H + K are both subgroups.
A typical element of HK is hk.

(1) Suppose $h_1k_1, h_2k_2 \in HK$
Then $(h_1k_1)(h_2k_2) = (h_1h_2)(k_1k_2)$
because G is abelian
and $h_1h_2 \in H$, $k_1k_2 \in K$ because both H and K are subgroups.

(2) $e_G \in H$, $e_G \in K$ since H and K are subgroups, so $e_G \cdot e_G = e_G \in HK$ (!)

(3) If $h \in H$, $k \in K$, then $h^{-1} \in H$ and $k^{-1} \in K$, so $h^{-1} \cdot k^{-1} \in HK$
But $h^{-1} \cdot k^{-1} = k^{-1} \cdot h^{-1} = (hk)^{-1}$
$\uparrow$ G is abelian.

So HK has inverses.

1b 1.8 - 2 $\tau^2\sigma$ (using σ, τ, μ from the first column.

τ^2. $\quad 1 \to 2 \to 4$ $\quad 3 \to 1 \to 2$ $\quad 5 \to 6 \to 5$
$\quad 2 \to 4 \to 3$ $\quad 4 \to 3 \to 1$ $\quad 6 \to 5 \to 6$

$\tau^2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 5 & 6 \end{pmatrix}$ $\quad \sigma: \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$

$\tau^2\sigma$: $\quad 1 \to 3 \to 2$ $\quad 3 \to 4 \to 1$ $\quad 5 \to 6 \to 6$ $\quad$ RIGHT TO LEFT
$\quad 2 \to 1 \to 4$ $\quad 4 \to 5 \to 5$ $\quad 6 \to 2 \to 3$

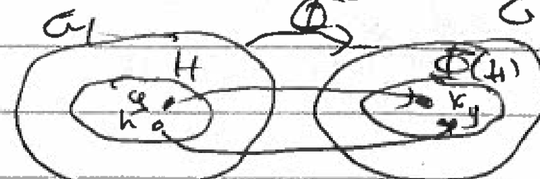
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 5 & 6 & 3 \end{pmatrix}$ ~~or~~ $(1\,2\,4\,5\,6\,3)$

2. G is a group
$H = \langle a \rangle = \{a^j\}$
$K = \langle b \rangle = \{b^k\}$.

Suppose $a \in \langle b \rangle$ and $b \in \langle a \rangle$
Then there exist r, s so that
$a = b^r$ and $b = a^s$.
Now suppose $x \in H$. Then
$x = a^j$ for some j, so
$x = a^j = (b^r)^j = b^{rj} \in K$.
~~Since~~ Thus $H \subseteq K$.
Similarly, if $y \in K$, then
$y = b^k$ for some k, so
$y = b^k = (a^s)^k = a^{sk} \in H$.
Thus $K \subseteq H$
so $H \subseteq K$, $K \subseteq H \Rightarrow H = K$.

3. G is a group, H and K are subgroups that are not contained in each other.
Pick $a \in H$, $a \notin K$
and $b \notin H$, $b \in K$
To prove: $ab \notin H \cup K$.
If $ab \in H \cup K$, then $ab \in H$ or $ab \in K$.
Suppose $ab \in H$. Since $a \in H$ and H is a subgroup, $a^{-1} \in H$.
Since H is a subgroup,
$a^{-1}(ab) = (a^{-1}a)b = b \in H$,
which contradicts the assumption!
(Similarly if $ab \in K$.)
Thus $ab \notin H \cup K$, so $H \cup K$ is not closed under $*$ and isn't a subgroup

4. G_1 G_2
$\Phi: G_1 \to G_2$ isomorphism.
H is a subgroup of G_1.
$\Phi(H) = \{\Phi(h) : h \in H\}$ is a subset of G_2.
To prove it's a subgroup, we need closure, identity, inverse
(i) Suppose $x, y \in \Phi(H)$.
Then there exist $g, h \in H$
so that $x = \Phi(g)$, $y = \Phi(h)$



We want to show $x *_{G_2} y \in \Phi(H)$
But $g * h \in H$ since H is a subgroup,
so $\Phi(H)$ contains
$\Phi(g *_{G_1} h) = \underbrace{\Phi(g)}_{x} *_{G_2} \underbrace{\Phi(h)}_{y}$
Thus $x *_{G_2} y$ is the image of an element in G_1. ✓.
(ii) $e_{G_1} \in H$ because H is a subgroup so $\Phi(H)$ contains $\Phi(e_{G_1})$, which we know is e_{G_2} ✓
(iii) Suppose $x \in \Phi(H)$, so
$x = \Phi(g)$ for some $g \in H$
Since H is a subgroup, $g \in H \Rightarrow$
$g^{-1} \in H \Rightarrow \Phi(g^{-1}) \in \Phi(H)$.
But $g *_{G_1} g^{-1} = e_{G_1} \Rightarrow$
$X = \Phi(g) *_{G_2} \Phi(g^{-1}) = e_{g_2}$, so
that $\Phi(g^{-1}) = x^{-1} \in \Phi(H)$ ✓

5. §1.8-20

$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$ $\rho = (124)(35)$

I'll only do ρ^2 in detail:

ρ^2: $1 \to 2 \to 4$, $2 \to 4 \to 1$, $3 \to 5 \to 3$, $4 \to 1 \to 2$, $5 \to 3 \to 5$

$\rho^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{pmatrix} = (142)(3)(5)$

$\rho^3 = \rho \cdot \rho^2$. $1 \to 4 \to 1$, $2 \to 1 \to 2$, $3 \to 3 \to 5$, $4 \to 2 \to 4$, $5 \to 5 \to 3$

$\rho^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix} = (1)(2)(4)(35)$

$\rho^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix} = (124)(3)(5)$

$\rho^5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix} = (142)(35)$

$\rho^6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = (1)(2)(3)(4)(5)$

	ρ^0	ρ^1	ρ^2	ρ^3	ρ^4	ρ^5
ρ^0	ρ^0	ρ^1	ρ^2	ρ^3	ρ^4	ρ^5
ρ^1	ρ^1	ρ^2	ρ^3	ρ^4	ρ^5	ρ^0
ρ^2	ρ^2	ρ^3	ρ^4	ρ^5	ρ^0	ρ^1
ρ^3	ρ^3	ρ^4	ρ^5	ρ^0	ρ^1	ρ^2
ρ^4	ρ^4	ρ^5	ρ^0	ρ^1	ρ^2	ρ^3
ρ^5	ρ^5	ρ^0	ρ^1	ρ^2	ρ^3	ρ^4

This, my friends, is a cyclic group of order 6 and is <u>not</u> isomorphic to S_3.

6. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} = (13)(245)$

$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix} = (1)(2345)$

Compute $\sigma\tau$ and $\tau\sigma$.

$\sigma\tau$: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$

$1 \to 1 \to 3$, $2 \to 3 \to 1$, $3 \to 4 \to 5$, $4 \to 5 \to 2$, $5 \to 2 \to 4$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$ or (13542)

$\tau\sigma$ $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$

$1 \to 3 \to 4$, $2 \to 4 \to 5$, $3 \to 1 \to 1$, $4 \to 5 \to 2$, $5 \to 2 \to 3$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$ or (14253)

To save space, I will write the powers of σ in cycle form. They resemble #5!

$\sigma = (13)(245)$

$\sigma^2 = (13)^2(245)^2 = (1)(3)(254)$

(actually (13) and (245) commute, so

$\sigma^2 = (13)(245)(13)(245) = (13)(13)(245)(245)$)

$\sigma^3 = (13)^3(245)^3 = (13)(2)(4)(5)$

so $\sigma^3 = (13)(2)(4)(5)$

$\sigma^4 = \sigma^3 \cdot \sigma = (13)(13)(245)$
$= (245)$

$\sigma^5 = \sigma^3 \cdot \sigma^2 = (13)(1)(3)(254)$
$= (13)(254)$

$\sigma^6 = \sigma^3 \cdot \sigma^3 = (13)(13) = (1)(3) = \sigma^0$.

so $k = 6$.

As noted in class: $(13)^2 = (1)(3)$, $(245)^3 = (2)(4)(5)$ and $6 = \mathrm{lcm}(2,3)$.