Paired with itself; not is $x \neq e$ so that $x \neq e$.

What This means is that

For $x$ such that $x^2 = e$, if $10$ is even, $G$ has a subgroup of order $2$. (Last sentence is a remark, not part of solution)

1b. 6/15 - 47

Suppose $G$ is an abelian group and $H = \{ x : x^2 = e \}$.

We need to prove that $H$ is a subgroup.

1) Is it closed? $x \in H$, $y \in H$ implies

$x^2 = e$, $y^2 = e$. What about

$xy$? $(xy)^2 = (xy)(xy) = x(yx)y = x(xy)y = x^2y^2 = e$. (\checkmark)

2) Is the identity there? $e^2 = e$ (\checkmark).

3) If $x \in H$, is $x^{-1} \in H$?

$x^2 = e \Rightarrow (x^{-1})^2 = x^{-1}x = x^{-1}$, so

$(x^{-1})^2 = x^2 = e$. (\checkmark)

All conditions are satisfied.

Remark: Note that for the nonabelian group $S_3$, it might not happen that $e$ is a subgroup. The square of any flip is $e$, but the product of the different flips is a rotation, which doesn't belong in $H$.

2. $\mathbb{Z}/12$,

$x = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \}$

We saw this on HW 1.

You showed that $\mathbb{Z}/12$ is a cyclic group, with generators $[3]_{12}$.

In $[3]_{12}$, $[27]_{12}$ is cyclic, all subgroups are cyclic.
keeping these in mind, the subgroups are:

\[ \{1/4, 1\} \cup \{1/3, 2/3, 1\} \cup \{1/2, 1/2, 1\} \cup \{1/6, 5/6, 1\} \]

The identity alone

\[ \{1/3, 1/3, 1/3\} \]

generated by \[ [1/3] = [3] \]

cut \[ \{1/3, 1/3, 1/3\} \]

generated by \[ [1/3] = [3] \]

3. Duplicate - Free Problem

4. \( G = \langle a \rangle \), \( a^{15} = 1 \).

We have a theorem that says:

If \( H = \langle a^k \rangle = \langle a \rangle \)

is a subgroup of \( G \) of order \( a \).

- \( k = 7 \), \( \text{gcd}(7, 15) = 1 \), so \( a^7 \) is a subgroup of \( G \) of order \( 7 \). (Check: \( 4 \cdot a^7 = a^{14} = a^{15} = 1 \).

b) \( k = 5 \), \( \text{gcd}(5, 15) = 1 \), so \( a^5 \) is a subgroup of \( G \) of order \( 5 \).

5a. \( \Phi \) is an isomorphism, so \( \Phi(1/3, 1/3, 1/3) = \Phi(1/3) \ast \Phi(1/3) \ast \Phi(1/3) \)

... (continued on the next page)