1. (E) Suppose \( G \) is a group and \( H \) and \( K \) are subgroups of \( G \). Prove that \( H \cap K \) is a subgroup of \( G \). (It is possible that \( H \cap K = \{e\} \).)

What we need to prove for a subgroup is that it is closed under the operation, that the identity exists and that inverses exist. Since a subgroup is a subset of a group, associativity is automatic. Suppose \( x \cdot y \in H \cap K \), since \( H \) and \( K \) are both groups, \( x \cdot y \in H \) and \( x \cdot y \in K \), so by the definition of \( \cap \), \( x \cdot y \in H \cap K \). Similarly, \( e_G \in H, K \), so \( e_G \in H \cap K \). Finally, if \( x \in H \cap K \), then \( x^{-1} \in H \) and \( x^{-1} \in K \), so \( x^{-1} \in H \cap K \).

2/3 (Counts as two problems) (E) Let \( G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \); the operation in \( G \) is addition.) Let \( H = \langle a \rangle, a^{12} = e \), be a cyclic group of order 12. Let \( \phi \) be a homomorphism from \( G \) to \( H \) defined by

\[
\phi([i]_2, [j]_4) = a^{6i+3j}; \quad 0 \leq i \leq 1, \quad 0 \leq j \leq 3.
\]

a. Compute \( \phi \) for the eight elements of \( G \).

b. Determine \( K = \text{Ker}(\phi) \subseteq G \) and \( \text{Im}(\phi) \subseteq H \).

c. Write down the cosets of \( K \) in \( G \).

d. Give the multiplication table of the factor group \( G/K \).

a. From the definition, and reducing by \( a^{12} = e \),

\[
\phi([0]_2, [0]_4) = e, \quad \phi([0]_2, [1]_4) = a^3, \quad \phi([0]_2, [2]_4) = a^6, \quad \phi([0]_2, [3]_4) = a^9, \\
\phi([1]_2, [0]_4) = a^6, \quad \phi([1]_2, [1]_4) = a^9, \quad \phi([1]_2, [2]_4) = e, \quad \phi([1]_2, [3]_4) = a^3.
\]

b. From the data in a., \( K = \text{Ker}(\phi) = \{([0]_2, [0]_4), ([1]_2, [2]_4)\} \) and \( \text{Im}(\phi) = \{e, a^3, a^6, a^9\} \).

c. By adding elements in the usual way, the cosets of \( K \) are:

\[
\{([0]_2, [0]_4), ([1]_2, [2]_4)\}, \quad \{([0]_2, [1]_4), ([1]_2, [3]_4)\}, \\
\{([0]_2, [2]_4), ([1]_2, [0]_4)\}, \quad \{([0]_2, [3]_4), ([1]_2, [1]_4)\}.
\]

Note that these are the inverse images of \( e, a^3, a^6, a^9 \) in that order.

d. So \( G/K \) consists of these four cosets, which I will call \( H_0, H_1, H_2 \) and \( H_3 \) for short. The multiplication table is found by the definition. I will show \( H_1 \star H_2 = H_3 \):

\[
\{([0]_2, [1]_4), ([1]_2, [3]_4)\} = \{([0]_2, [2]_4), ([1]_2, [0]_4)\} + \{([0]_2, [0]_4), ([1]_2, [2]_4)\} \\
= \{([0]_2, [3]_4), ([1]_2, [1]_4), ([1]_2, [5]_4), ([2]_2, [3]_4)\} \\
= \{([0]_2, [3]_4), ([1]_2, [1]_4)\}
\]

Here’s the table which shows that \( G/K \) is isomorphic to \( \text{Im}(\phi) \), which is a cyclic group of order 4.

<table>
<thead>
<tr>
<th>( K/G )</th>
<th>( H_0 )</th>
<th>( H_1 )</th>
<th>( H_2 )</th>
<th>( H_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_0 )</td>
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<td>( H_1 )</td>
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</tr>
</tbody>
</table>
4. (E) Suppose $G$ is an abelian group and let $L = G \times G = \{(g_1, g_2) \mid g_1, g_2 \in G\}$, with the usual operation on $L$. Define a map $\phi : L \mapsto G$ by 

$$\phi(g_1, g_2) = g_1^2g_2.$$ 

a. Prove that $\phi$ is a homomorphism from $L$ to $G$.
b. For $g \in G$, prove that there is a unique $h \in G$ so that $(g, h) \in \text{Ker}(\phi)$, and describe $h$ in terms of $g$.

a. The solution is a matter of following the letters. Let’s take two elements in $L$: $(g_1, g_2)$ and $(g_3, g_4)$. Then, by definition

$$L(g_1, g_2) = g_1^2g_2 \quad L(g_3, g_4) = g_3^2g_4.$$ 

By the definition of $L = G \times G$, we have

$$(g_1, g_2) \ast_G (g_3, g_4) = (g_1g_3, g_2g_4) \implies \phi((g_1, g_2) \ast_G (g_3, g_4)) = \phi(g_1g_3, g_2g_4)$$

$$= (g_1g_3)^2(g_2g_4) = g_1g_3g_1g_3g_2g_4.$$ 

So we need to verify that

$$L(g_1, g_2) \ast_L L(g_3, g_4) = L((g_1, g_2) \ast_G (g_3, g_4)) \iff (g_1^2g_2)(g_3^2g_4) = g_1g_3g_1g_3g_2g_4,$$

but this equality follows from the hypothesis that $G$ is abelian.
b. There may be less here than meets the eye:

$$(g, h) \in \text{Ker}(\phi) \iff \phi(g, h) = e_G \iff g^2h = e_G \iff h = g^{-2}.$$ 

5. (E) Suppose $K = \{e, a, b\}$ is a normal subgroup of a group $G$. Here, I mean that $e$, $a$ and $b$ are different elements. Suppose $g \in G$ has the property that $ga \neq ag$. Prove that $ga = bg$.

(Hints: (i) Don’t panic! (ii) What does it mean for $K$ to be a normal subgroup?)

Since $K$ is a normal subgroup, we have in particular that $gK = Kg$. This means that 

$\{ge, ga, gb\} = \{g, ga, gb\}$ and $\{eg, ag, bg\} = \{g, ag, bg\}$ are equal as sets. In particular, $ga \in \{g, ag, bg\}$. We are told that $ga \neq ag$, and if $ga = g$, then $a = e$, but we’re told that $a \neq e$. This leaves us no choice: we must have $ga = bg$. 

Ungraded problems

6. Write the following two permutations in $S_9$ in the cycle notation.

\[
\pi_1 = \left( \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
5 & 6 & 1 & 9 & 2 & 3 & 4 & 8 & 7 \\
\end{array} \right) \quad \pi_2 = \left( \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
3 & 6 & 5 & 2 & 9 & 8 & 1 & 7 & 4 \\
\end{array} \right)
\]

Also, compute $\pi_1 \pi_2$, making sure you choose the correct order.

Following the elements, $\pi_1 = (15263)(497)(8)$ and $\pi_2 = (135942687)$. To find $\pi_1 \pi_2$, follow the elements.

\[
1 \rightarrow 3 \rightarrow 1, \quad 2 \rightarrow 6 \rightarrow 3, \quad 3 \rightarrow 5 \rightarrow 2, \quad 4 \rightarrow 2 \rightarrow 6, \quad 5 \rightarrow 9 \rightarrow 7, \\
6 \rightarrow 8 \rightarrow 8, \quad 7 \rightarrow 1 \rightarrow 5, \quad 8 \rightarrow 7 \rightarrow 4, \quad 9 \rightarrow 4 \rightarrow 9.
\]

Thus,

\[
\pi_1 \pi_2 = \left( \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 3 & 2 & 6 & 7 & 8 & 5 & 4 & 9 \\
\end{array} \right) = (1)(23)(468)(57)(9).
\]

7. Someone tells you, accurately, that $((\mathbb{Z}/19\mathbb{Z})^*, \circ)$ is a cyclic group of order 18, and that one generator is $[14]_{19}$. Express all the generators of this group in the form $[14^k]_{19}$. If you take any numerical powers of 14, then you are working much too hard! Note: nobody told me there was a typo in the printed version! I meant $[14]_{19}$ not $[14]_{18}$!

Since $((\mathbb{Z}/19\mathbb{Z})^*, \circ)$ is a cyclic group of order 18, with generator $[14]_{19}$, every element of the group can be written in the form $[14^k]_{19}$ for $k \in \{0, \ldots, 17\}$. By a class theorem, the order of this element is $18/\gcd(k, 18)$, so $[14^k]_{19}$ is a generator if and only if $\gcd(k, 18) = 1$. Since $18 = 2 \cdot 3^2$, this means that $k$ is not divisible by 2 or 3. The choices in $\{0, \ldots, 17\}$ are $k \in \{1, 5, 7, 11, 13, 17\}$, so the generators are

$[14]_{19}, \ [14^5]_{19}, \ [14^7]_{19}, \ [14^{11}]_{19}, \ [14^{13}]_{19}, \ [14^{17}]_{19}$


8. Define the binary operator $*$ on the set of positive real numbers by $a * b = \sqrt{ab}$. Is $*$ associative?

Let’s check, and use exponent notation, so $a * b = a^{1/2}b^{1/2}$. Then

\[
(a * b) * c = a^{1/2}b^{1/2} * c = a^{1/4}b^{1/4}c^{1/2} \\
a * (b * c) = a * b^{1/2}c^{1/2} = a^{1/2}b^{1/4}c^{1/4}
\]

These are not in general equal. We have

\[
\frac{(a * b) * c}{a * (b * c)} = (c/a)^{1/4},
\]

so if $c \neq a$, then $(a * b) * c \neq a * (b * c)$. It follows that $*$ is not associative.