It’s ok to work together, but don’t copy without comprehension. The symbol ($E$) means that at least part of this problem is a former exam question.

1. ($E$) Consider the integers $\mathbb{Z}$ with the binary operation $\circ$ defined by $a \circ b = a + b - 4$.

   a. Prove that $\circ$ is associative.
   b. Find the identity element for $\circ$.
   c. Find an inverse element (with respect to $\circ$) for $a \in \mathbb{Z}$.

2. ($E$) Let $G = (\mathbb{Z}/6\mathbb{Z}, \oplus)$ and $H = ((\mathbb{Z}/9\mathbb{Z})^*, \odot)$. Write down one explicit isomorphism $\Phi$ from $G$ to $H$; that is, determine the values of $\Phi([0]_6)$, $\Phi([1]_6)$, $\Phi([2]_6)$, $\ldots$ as elements in $H$. Your answer should include an explanation of why you know that $\Phi$ is an isomorphism.

3. and 4. ($E$) (counts as two problems) Define in the usual way the set $S = (\mathbb{Z}/5\mathbb{Z})^* = \{[1]_5, [2]_5, [3]_5, [4]_5\}$, and define the binary operation $\ast$ on $S$ by $[a]_5 \ast [b]_5 = [ab]_5$. (For example $[1]_5 \ast [4]_5 = [2 \cdot 1 \cdot 4]_5 = [8]_5 = [3]_5$. ) For your convenience, the multiplication table is given above. (The table shows that $\ast$ is a binary operation, and you do not have to prove this!) Instead, your task in this problem is to prove that $G = (S, \ast)$, with this strange definition, is a group.

   a. Show that $\ast$ is associative in $G$.
   b. Determine the identity element in $(S, \ast)$. (The table will be helpful.)
   c. Find the inverses of the four elements in $S$. (I want four answers here.)
   d. Write down enough powers of $[1]_5$ to show that $(S, \ast)$ is a cyclic group and explain your answer.

Don’t forget page 2!
5. (Followup to 9/4 Worksheet) Let \( G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \{([a]_2, [b]_4)\} \), and define the operation \(*\) by

\[
([a]_2, [b]_4) * ([c]_2, [d]_4) = ([a + c]_2, [b + d]_4).
\]

Thus, for example,

\[
([1]_2, [2]_4) * ([1]_2, [3]_4) = ([2]_2, [5]_4) = ([0]_2, [1]_4),
\]

because \(2 \equiv 0 \pmod{2}\) and \(5 \equiv 1 \pmod{4}\).

a. Determine \(\langle ([1]_2, [1]_4) \rangle\), \(\langle ([1]_2, [2]_4) \rangle\) and \(\langle ([1]_2, [3]_4) \rangle\).

b. Show that \(H = \{([0]_2, [0]_4), ([0]_2, [2]_4), ([1]_2, [0]_4), ([1]_2, [2]_4)\}\) is a subgroup for \(G\), and give an isomorphism from \(H\) to the Klein 4-group \(V\).