Math 417 **Homework 2** **Due Friday, September 11, 2020**

It's ok to work together, but don't copy without comprehension. The symbol ($\mathcal{E}$) means that at least part of this problem is a former exam question.

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1. ($\mathcal{E}$) Consider the integers $\mathbb{Z}$ with the binary operation $\circ$ defined by

$$a \circ b = a + b - 4.$$

We want to prove that $(\mathbb{Z}, \circ)$ is a group.

a. Prove that $\circ$ is associative.
b. Find the identity element for $\circ$.
c. Find an inverse element (with respect to $\circ$) for $a \in \mathbb{Z}$.

2. ($\mathcal{E}$) Let $G = (\mathbb{Z}/6\mathbb{Z}, \oplus)$ and $H = ((\mathbb{Z}/9\mathbb{Z})^*, \odot)$. Write down one explicit isomorphism Φ from G to H; that is, determine the values of $\Phi([0]_6)$, $\Phi([1]_6)$, $\Phi([2]_6), \ldots$ as elements in H. Your answer should include an explanation of why you know that Φ is an isomorphism.

3. and 4. ($\mathcal{E}$) (counts as two problems) Define in the usual way the set

$$S = (\mathbb{Z}/5\mathbb{Z})^* = \{[1]_5, [2]_5, [3]_5, [4]_5\},$$

and define the binary operation $*$ on S by $[a]_5 * [b]_5 = [2ab]_5$. (For example $[1]_5 * [4]_5 = [2 \cdot 1 \cdot 4]_5 = [8]_5 = [3]_5$.) For your convenience, the multiplication table is given above. (The table shows that $*$ is a binary operation, and you do **not** have to prove this!) Instead, your task in this problem is to prove that $G = (S, *)$, with **this** strange definition, is a group.

G	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[1]_5$	$[2]_5$	$[4]_5$	$[1]_5$	$[3]_5$
$[2]_5$	$[4]_5$	$[3]_5$	$[2]_5$	$[1]_5$
$[3]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[4]_5$	$[3]_5$	$[1]_5$	$[4]_5$	$[2]_5$

a. Show that $*$ is associative in G.
b. Determine the identity element in $(S, *)$. (The table will be helpful.)
c. Find the inverses of the four elements in S. (I want four answers here.)
d. Write down enough powers of $[1]_5$ to show that $(S, *)$ is a cyclic group and explain your answer.

Don't forget page 2!

5. (Followup to 9/4 Worksheet) Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \{([a]_2, [b]_4)\}$, and define the operation $*$ by

$$([a]_2, [b]_4) * ([c]_2, [d]_4) = ([a+c]_2, [b+d]_4).$$

Thus, for example

$$([1]_2, [2]_4) * ([1]_2, [3]_4) = ([2]_2, [5]_4) = ([0]_2, [1]_4),$$

because $2 \equiv 0 \mod 2$ and $5 \equiv 1 \mod 4$.

a. Determine $\langle([1]_2, [1]_4)\rangle$, $\langle([1]_2, [2]_4)\rangle$ and $\langle([1]_2, [3]_4)\rangle$.

b. Show that $H = \{([0]_2, [0]_4), ([0]_2, [2]_4), ([1]_2, [0]_4), ([1]_2, [2]_4)\}$ is a subgroup for G, and give an isomorphism from H to the Klein 4-group V.